BOWLAND MATHS

IMPULS

Nuffield Foundation

The University of Nottingham

UNITED KINGDOM · CHINA · MALAYSIA

Lesson Study for Mathematical Problem Solving

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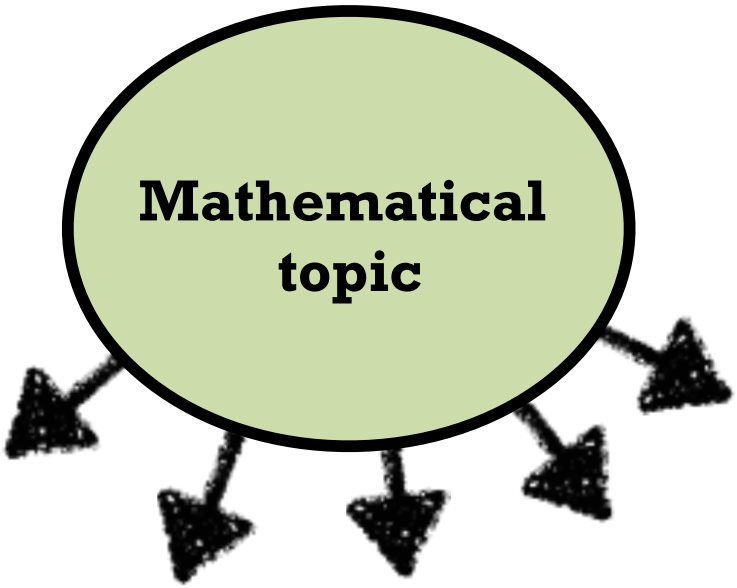
AMET 2015

Problem Solving

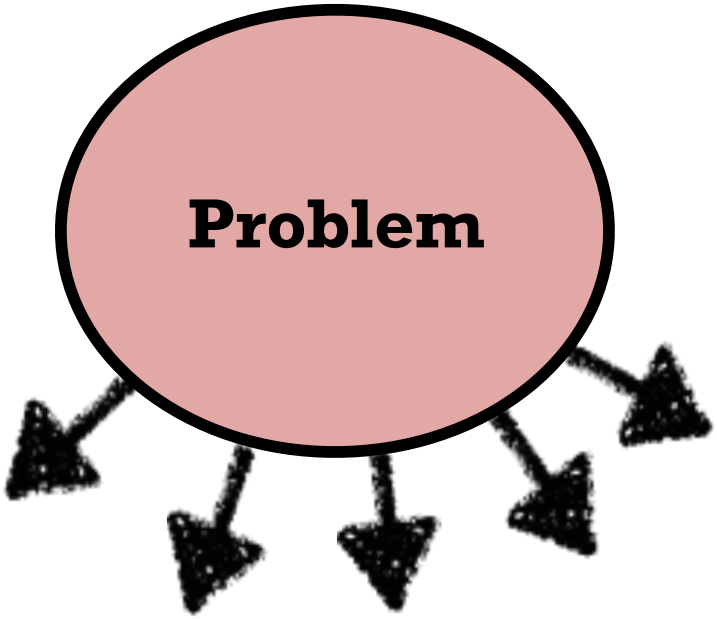
Problem solving is the process of tackling unstructured problems that require students to model situations with mathematics, make reasoned assumptions, construct chains of reasoning and interpret solutions in context.

Concept focused lesson



Illustrative Applications

Problem solving focused lesson



Choose appropriate mathematical tools

Goal
Factual recall Procedural fluency
Conceptual understanding Reasoning and communicating
Solving problems Mathematical literacy

Three Strands of the NC for England

Develop fluency and conceptual understanding

- "become fluent in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately."

Reason mathematically

- "by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language"

Solve problems

- "by applying their mathematics to a variety of routine and non- routine problems with increasing sophistication, including breaking down problems into a series of simpler steps and persevering in seeking solutions."

Goal	Student Product	Task and Activity "Genres"
Factual recall Procedural fluency	• Performance	• Memorise and rehearse
Conceptual understanding Reasoning and communicating	• Classification	• Sort, classify, define and deduce
	• Representation	• Describe, interpret and translate
	• Analysis	• Explore structure, variation, connections
	• Argument	• Test, justify and prove conjectures
Solving problems Mathematical literacy	• Model	• Formulate situations and problems
	• Solution	• Employ strategies
	• Critique	• Interpret and evaluate solutions, strategies

This is in the new NC …

Formulate situations and problems

- "begin to **model** situations mathematically and express the results using a range of formal mathematical representations"

Employ strategies

- "**select** appropriate concepts, methods and techniques to apply to unfamiliar and non- routine problems."

Interpret and evaluate

- "develop their use of formal mathematical knowledge to **interpret** and solve problems, including in financial mathematics"
- "develop their mathematical knowledge, in part through solving problems and **evaluating** the outcomes, including multi-step problems"

GCSE (2015) Assessment Objectives

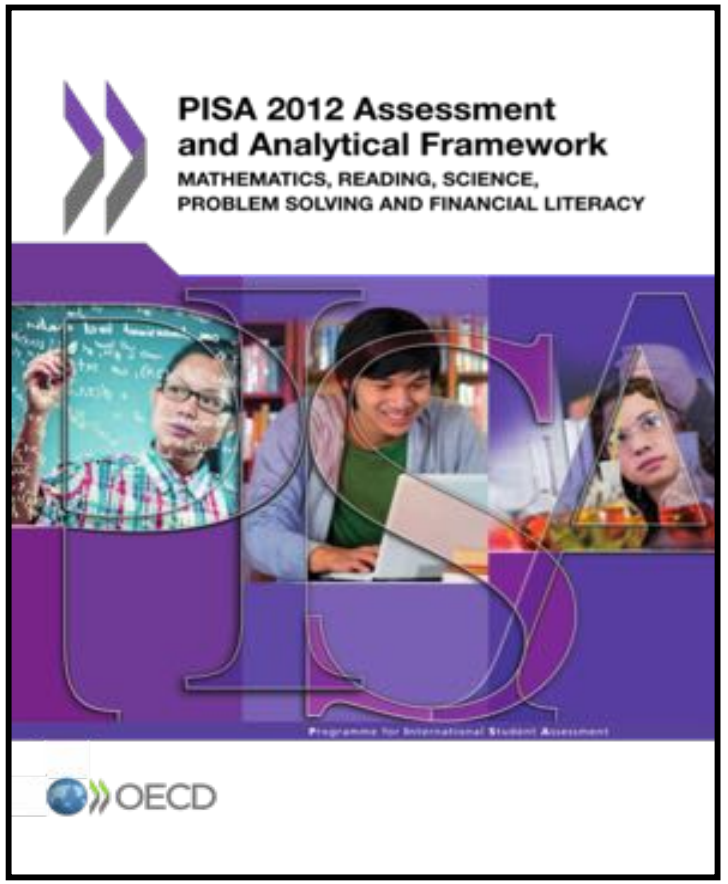
2015 Assessment Objectives		Weighting	
		Higher	Foundation
AO1	**Develop fluency and understanding** Use and apply standard techniques	40%	50%
AO2	**Reason and communicate** Reason, interpret and communicate mathematically	30%	25%
AO3	**Solve problems** Solve problems within mathematics and in other contexts	30%	25%

GCSE Assessment Objectives

AO3		Weighting
	Formulate • translate problems in mathematical or non-mathematical contexts into a process or a series of mathematical processes **Employ** • make and use connections between different parts of mathematics **Interpret** • interpret results in the context of the given problem **Evaluate** • evaluate methods used and results obtained • evaluate solutions to identify how they may have been affected by assumptions made.	30% (Higher) 25% (Foundation)

"Mathematical Literacy" in PISA is defined as:

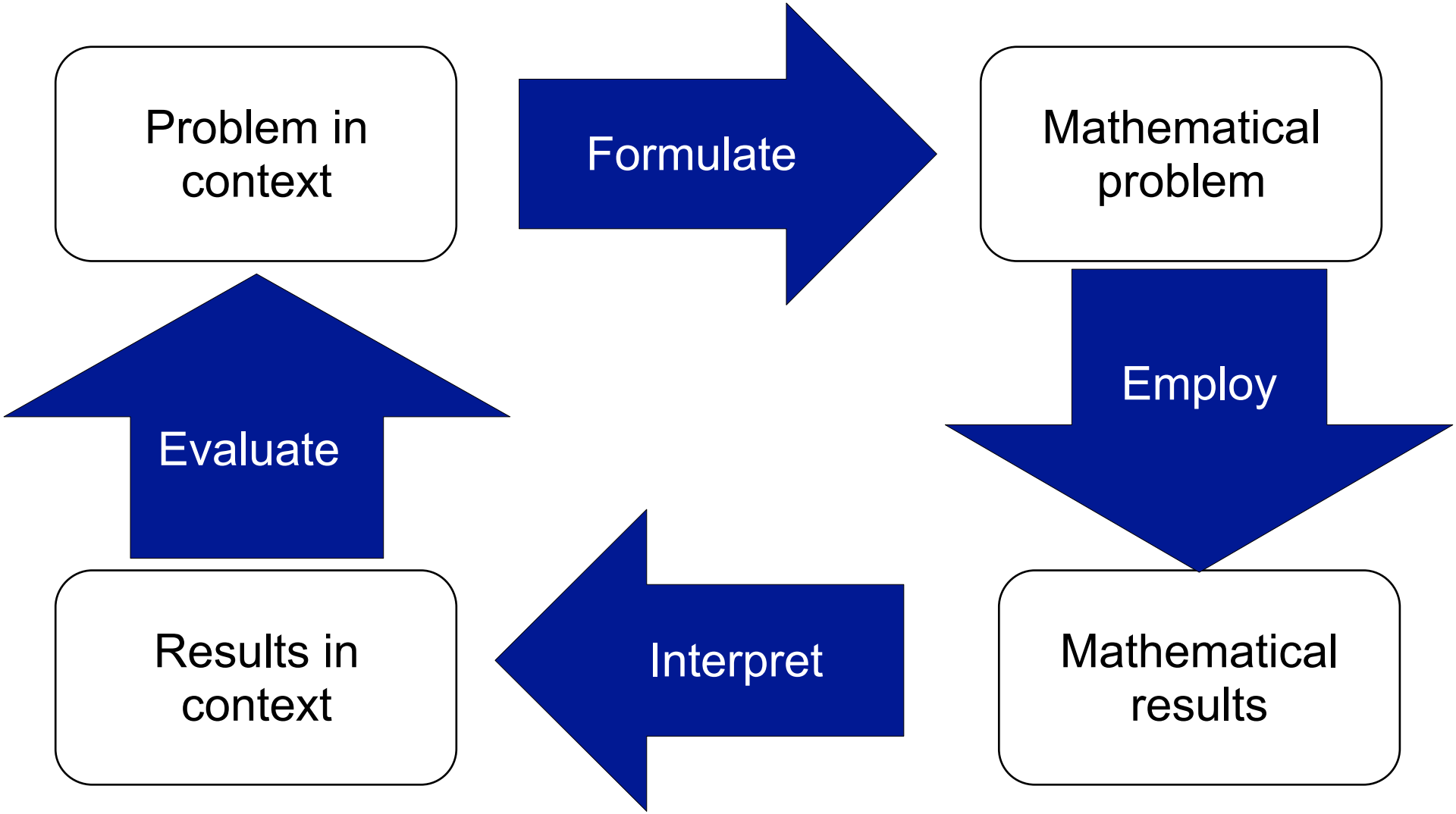
"An individual's capacity to **formulate, employ, and interpret** mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena."



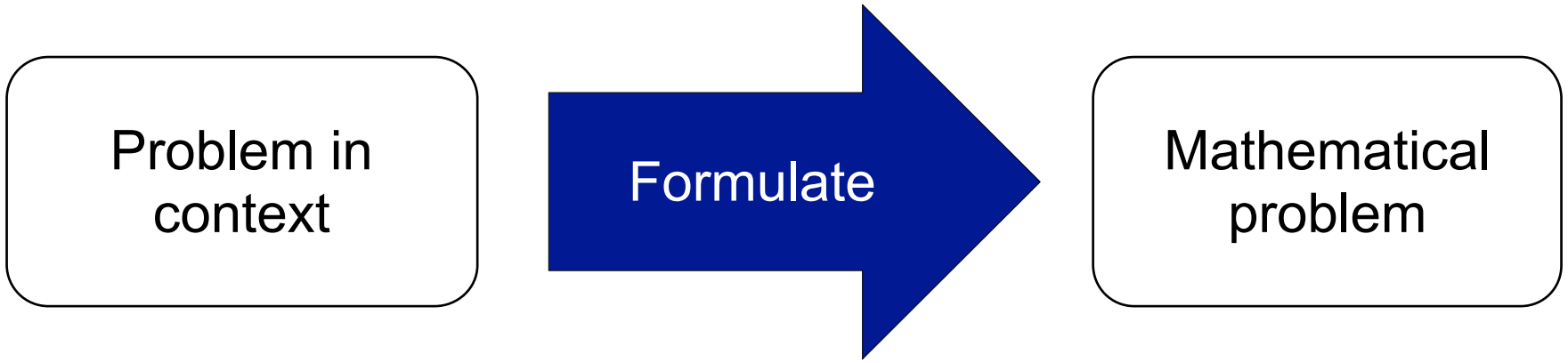
(PISA 2012, 2015 Mathematics Framework; OECD, 2013)

Mathematical literacy (PISA, 2015)

"The modelling cycle is a central aspect of the PISA conception of students as active problem solvers"



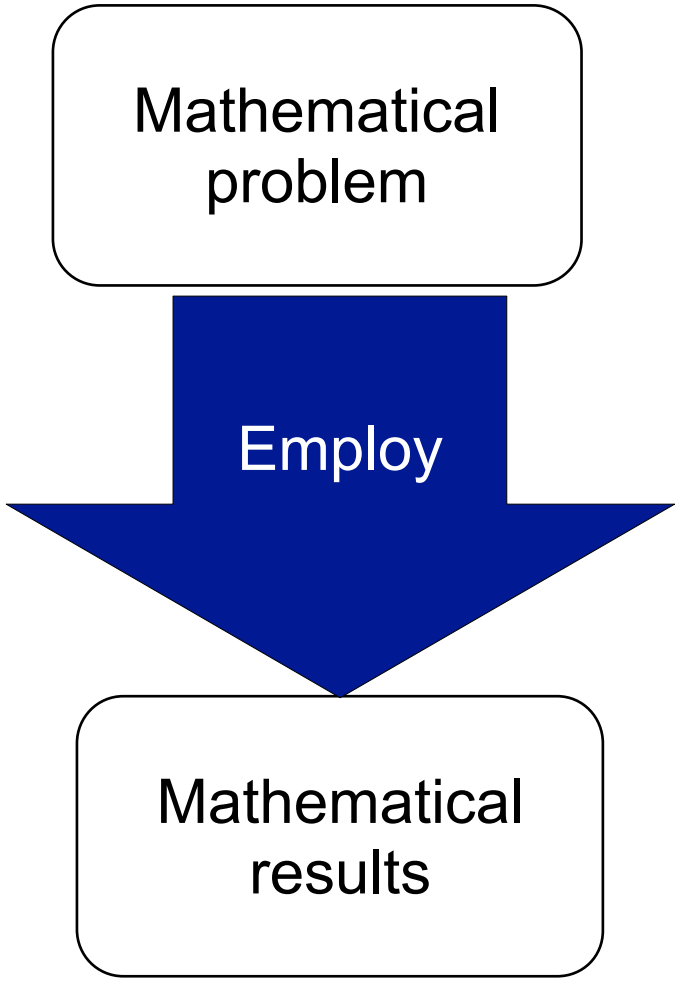
Formulating situations mathematically



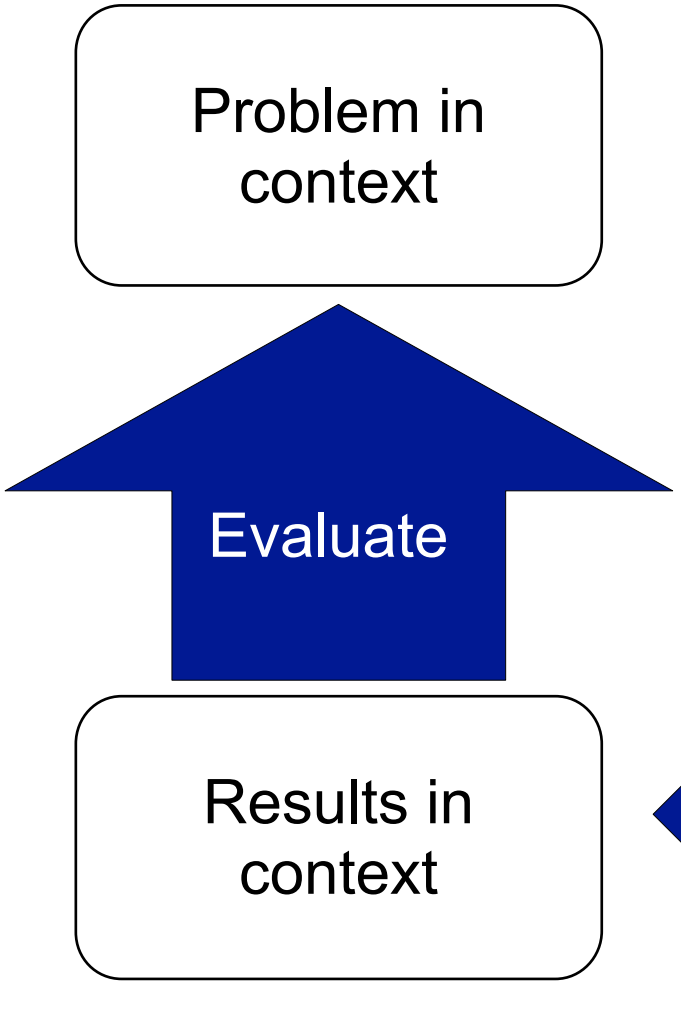
- Identify accessible questions that may be tackled
- Make suitable assumptions to simplify a situation
- Represent a situation mathematically
- Identify significant variables
- Generate relationships between variables
- Making connections with known problems or mathematical concepts, facts, or procedures

Employing concepts, facts, procedures and reasoning

- Select appropriate mathematical concepts and procedures
- Plan an approach
- Carry out the plan, monitoring progress and changing direction where necessary.



Interpreting and evaluating



- interpreting results in the real world context;
- evaluating the reasonableness of a mathematical solution in the context;
- explaining why a result or conclusion does, or does not, make sense in the context;
- identifying and critiquing the limits of the model.

Making and selling magazines

A group of bored, penniless teenagers want to make some money by producing and selling their own home-made magazine.

Generate questions / variables

- Who is the magazine for?
- What should it be about?
- How long should it be? — *l* pages
- How many writers? — *w* writers
- How long will it take to write? — *t* hours
- How much will it cost to produce? — *c* pence
- How many people will buy it? — *n* people
- What will be the selling price? — *s* pence
- How much profit will it make? — *p* pence
- How much should we spend on advertising? — *a* pence

Select variables

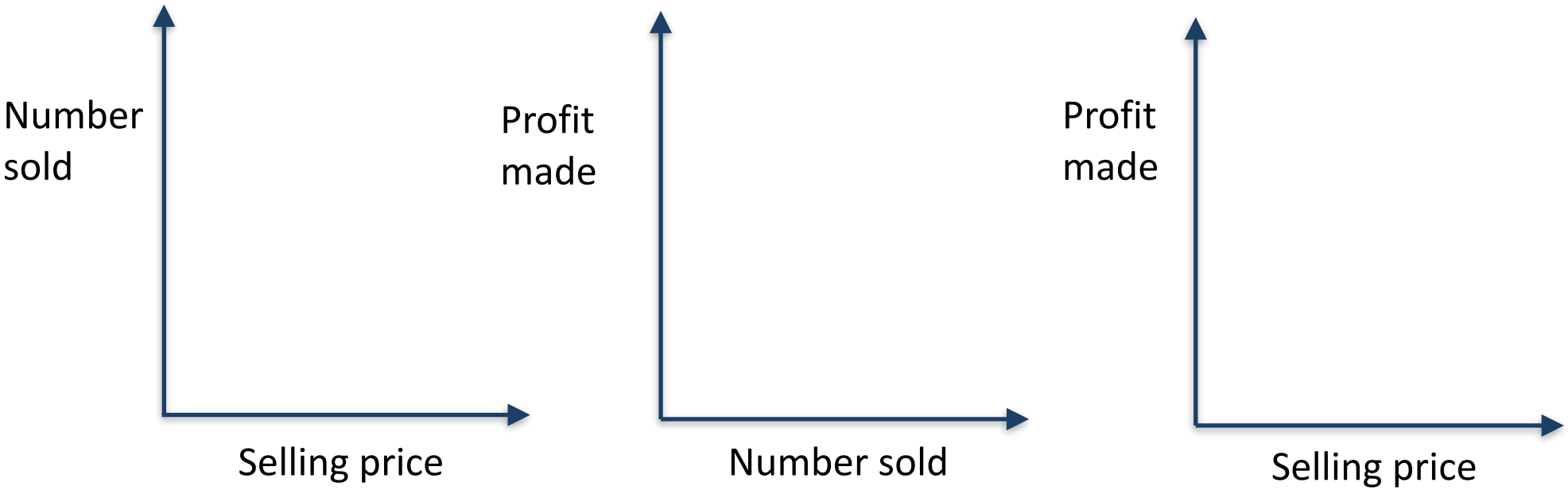
Which are relevant for calculating profit?

- **How many people will buy it?** — ***n* people**
- **What will be the selling price?** — ***s* pence**
- **How much profit will we make altogether?** — ***p* pence**
- **How much to produce each magazine?** — ***m* pence**
- **How much to advertise per week?** — ***a* pence**
- **How many weeks will we advertise for?** — ***w* weeks**

Generate relationships

How does:

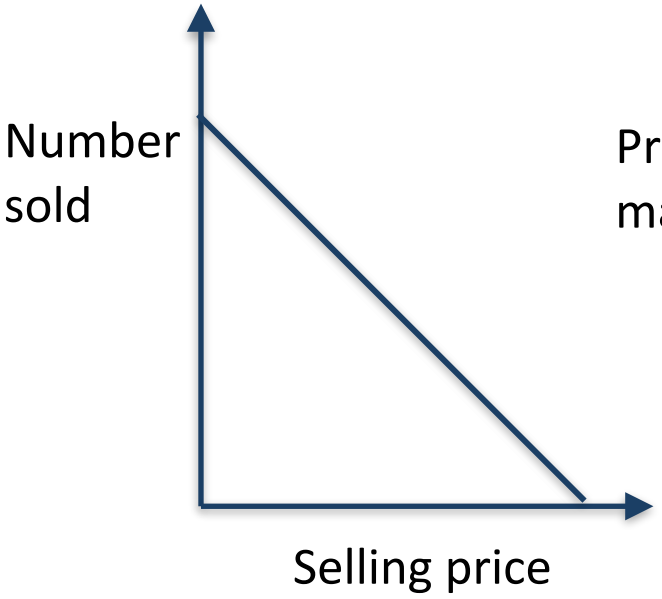
- **The number sold depend on the selling price?**
- **Profit made depend on number sold?**
- **The profit made depend on selling price?**

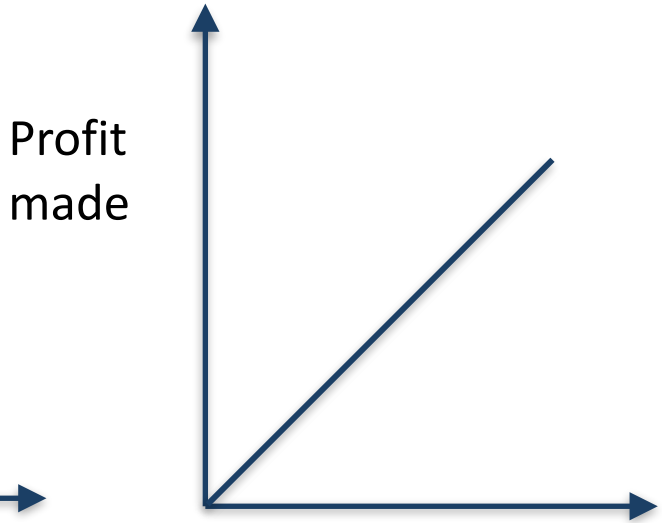


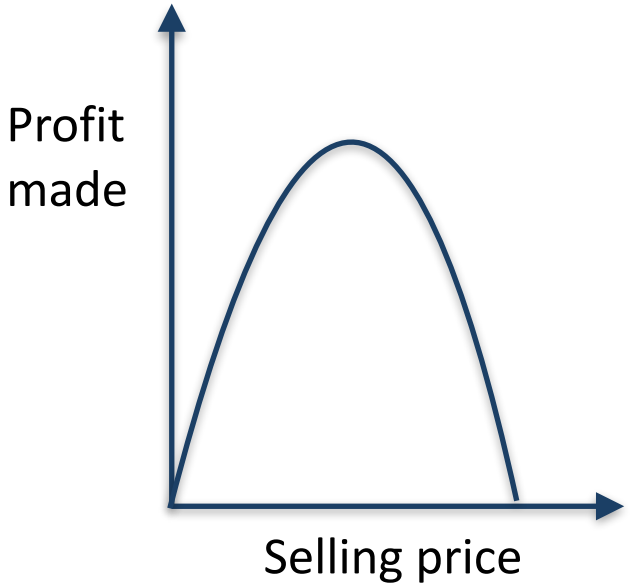
Generate relationships

How does:

- The number sold depend on the selling price?
- Profit made depend on number sold?
- The profit made depend on selling price?



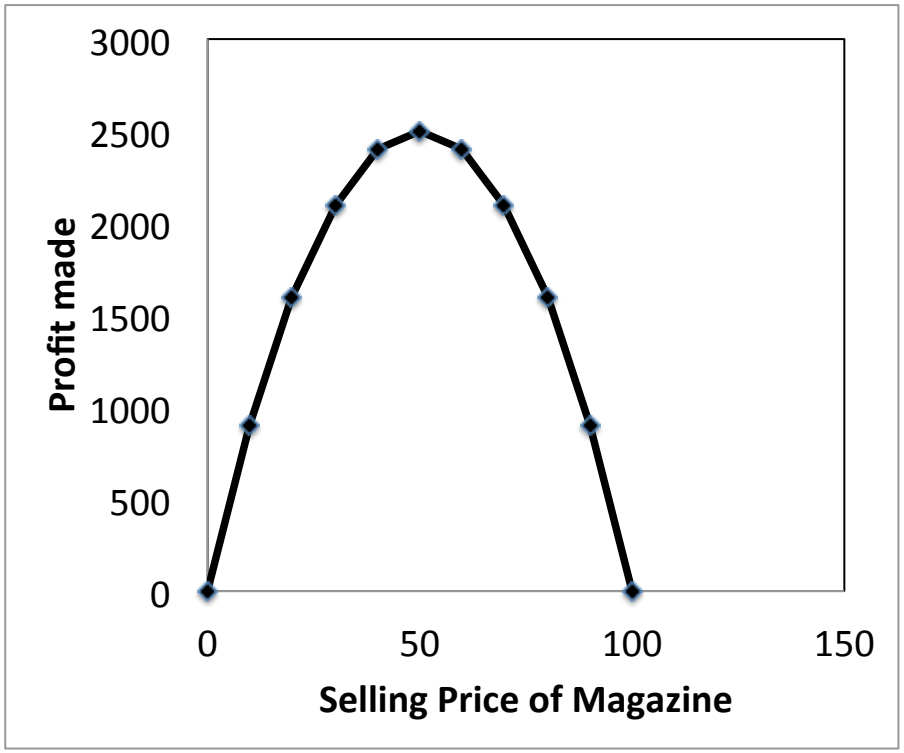




Make simplifying assumptions

- Production costs are free.
- Advertising will be by word of mouth only
- People cannot afford more than £1.
- If they are free we will get rid of 100 magazines.
- As price increases, fewer people will buy them.
- For every 10p increase, 10 fewer will buy them.

Employ the maths and interpret.



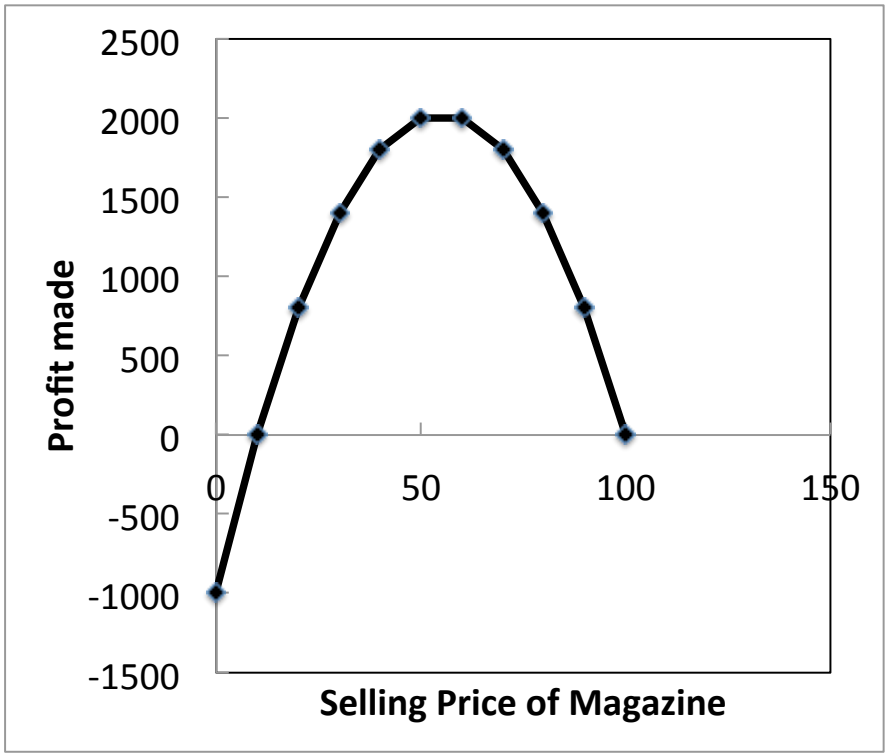
Profit = ***selling price*** **x** ***no. who buy***
= ***sn***
= ***s(100 - 10s)***

Optimum selling price = 50p
Profit made = £25

Revise assumptions

- Production costs are 10p per copy.
- Advertising will be by word of mouth only
- People cannot afford more than £1.
- If they are free we will get rid of 100.
- As price increases, fewer people will buy them.
- For every 10p increase 10 fewer will buy them.
- We wont waste magazines - we will print them ‘on demand’.

Revise assumptions



Profit = *selling price* x *no. who buy - prod. costs*

= *sn - 10n*

= *s(100 - s) - 10(100 - s)*

= *(s - 10)(100 - s)*

Optimum selling price = 55p

Profit made = £20.25

How many Dentists are there in the UK?

Formulating

Identifying significant variables and making assumptions

- Size of population	p	*60,000,000*
- Length of an appointment	t	*0.5 hours*
- Number of hours worked per day	n	*7 hours*
- Number of days worked per week	d	*5*
- Number of weeks worked per year	w	*40*
- Number of appointments per year	a	*2*

Derive relationships and facts

- People seen by dentist per day	$n \div t$	*14*
- Appointments per week	$d(n \div t)$	*70*
- Appointments per year	$dw\ (n \div t)$	*2800*
- Patients seen per year	$dw\ (n \div t) \div a$	*1400 =1500*
- Dentists	$p \div (dw\ (n \div t) \div a)$	*40,000*

Making reasonable estimates

There are about 60 million people in the UK.

About how many schoolteachers are there?

Formulating

Identifying significant variables and making assumptions

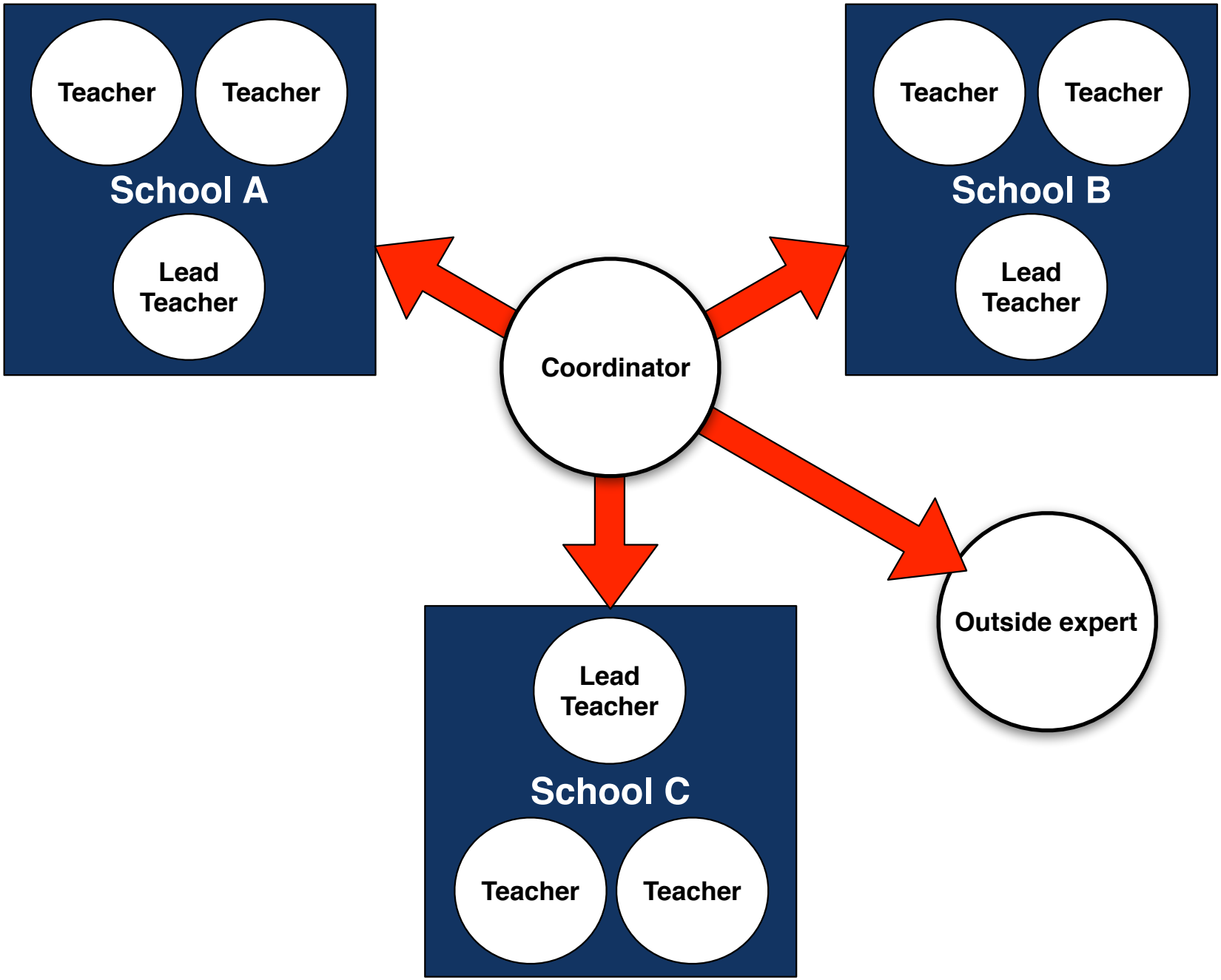
- Size of population	*p*	*60,000,000*
- How long do you go to school	*t*	*12 years*
- Average lifespan	*n*	*80 years*
- Size of class	*c*	*25*

Derive relationships and facts

- Fraction of pop. at school	*t ÷ n*	*1/7*
- School population	*p (t ÷ n)*	*8,500,000*
- Number of teachers	*p (t ÷ n) ÷ 25*	*340,000*

Lesson Study

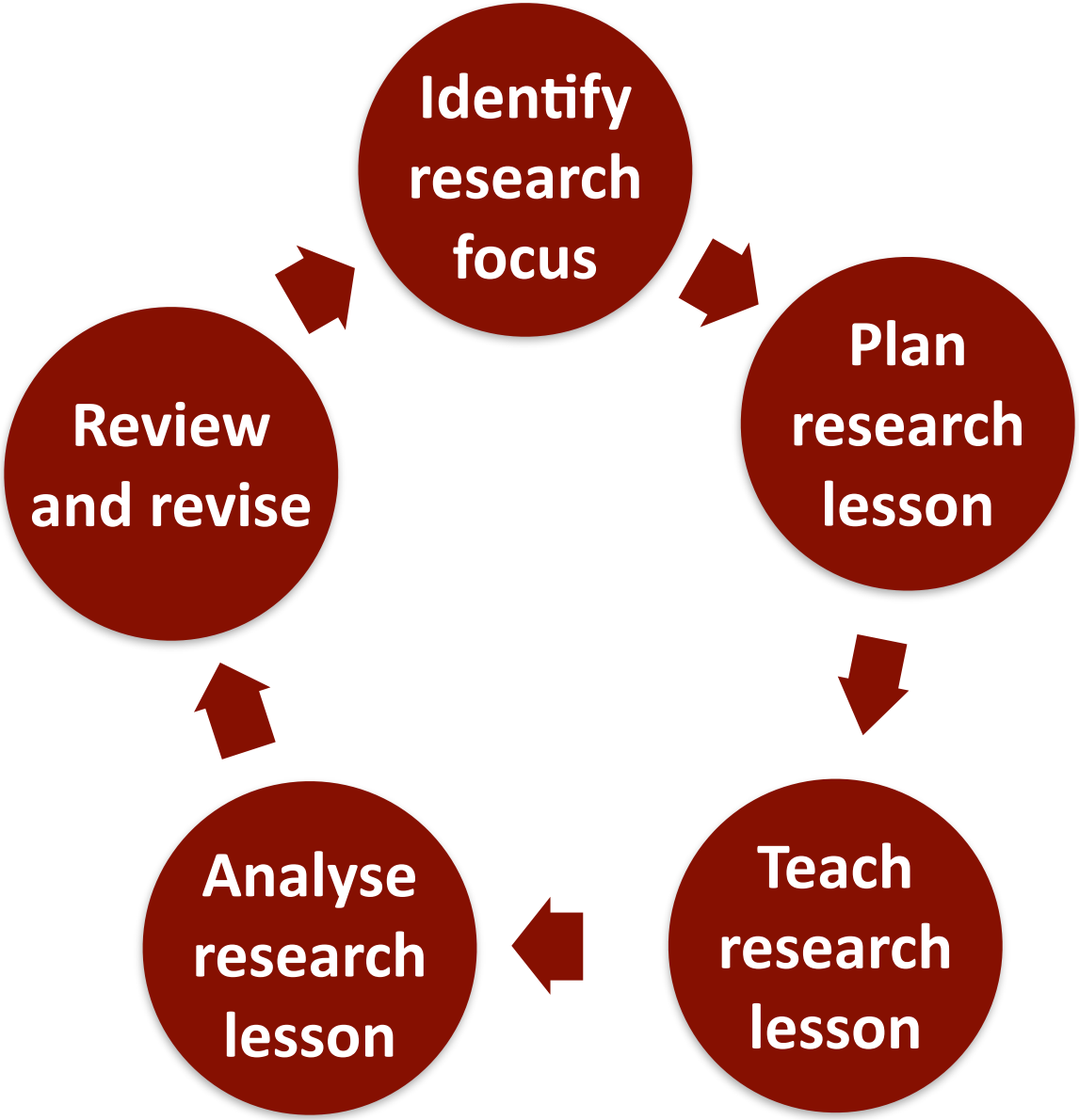
A professional learning community



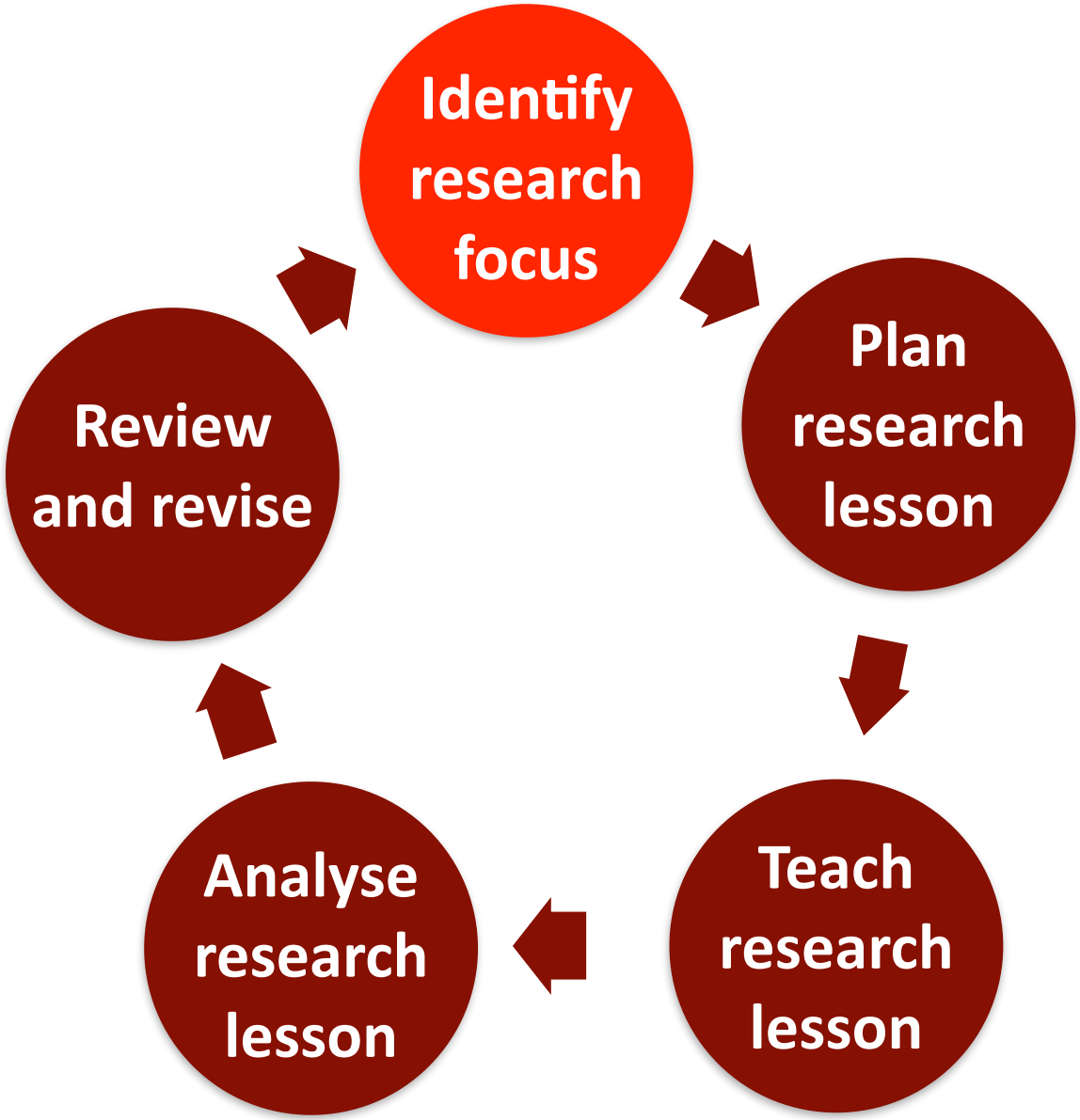
Research lessons (kenkjuu jugyou)

- **Lessons taught by normal teacher.**
- **They are focused on a significant research question**
  - on an issue that is of particular interest to the school or district; such as 'become active problem solvers'; 'be independent learners'.
- **They are collaboratively planned**
  - often over several months, and practiced with other classes
- **They are observed**
  - within school + few outside commentators, or teachers from a district, town , region or national.
- **They are recorded**
  - they may be video or audio recorded; student work and data from observations are collected.
- **They are analysed**
  - in post-lesson discussions.

Japanese Lesson Study Model



Japanese Lesson Study Model



To develop:

- **Ability to tackle unstructured problems.**
- **Enjoyment of Maths.**
- **Students' explanations.**
- **Effective student-student collaboration**

Identify the research focus

How might we help students to:

- formulate and pursue their own questions?
- select more powerful representations when problem solving?
- make sensible assumptions and analyse their effect on solutions?
- identify important variables in a problem and the relationships between them?
- become more systematic?
- plan approaches more carefully before embarking on them?
- compare the effectiveness of two or more different approaches to a problem?
- analyse problem solving strategies used by other students?
- communicate reasoning more effectively?

Example 1

- **How can we get students to look at the impact of making different assumptions?**

Example 1: A possible task

Pam is twenty today.
She is holding a party at which she plans to play the game 'Wrap the mummy'.
In this game, players try to completely cover themselves with toilet roll.

- Will one toilet roll be enough to wrap a person?
- Describe your reasoning as fully as possible.

Example 2

How can we help students to generate and select significant relationships between variables?

Example 2: A possible task

A music teacher runs a guitar class for 20 weeks. The class meets each week in a rented music studio.

It costs the teacher £400 to hire the studio for the whole course. The class contains 8 students. Each student pays the teacher a single fee of £70 for the course.

What profit does the teacher make?

Explore Structure, Variation and Connections

		s
The cost of hiring the studio	**£**	**400**
		n
Number of students in the class		**8**
		f
Fee for one student to attend the course	**£**	**70**
		p
Profit made by the teacher	**£**	**160**

$$p = 70 \times 8 - 400$$

$$p = fn - s$$

		s
The cost of hiring the studio	**£**	**400**
		n
Number of students in the class		**8**
		f
Fee for one student to attend the course	**£**	**70**
		p
Profit made by the teacher	**£**	

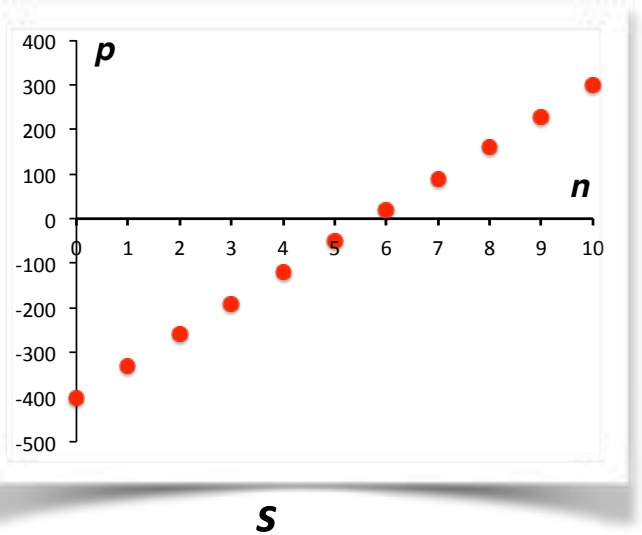
$$f = \frac{160 + 400}{8}$$

$$f = \frac{p + s}{n}$$

		s
The cost of hiring the studio	**£**	**400**
		n
Number of students in the class		**8**
		f
Fee for one student to attend the course	**£**	
		p
Profit made by the teacher	**£**	**160**

$$p = 70n - 400$$

n	0	2	4	6	8	10
p	-400	-260	-120	20	160	300



The cost of hiring the studio

s

£ 400

Number of students in the class

n

Fee for one student to attend the course

f

£ 70

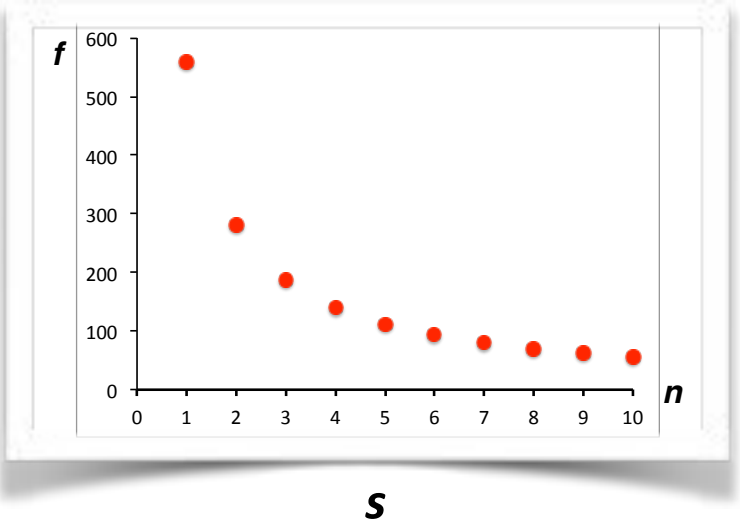
Profit made by the teacher

p

£

$$f = \frac{560}{n}$$

n	1	2	4	6	8	10
f	560	280	140	93	70	56



The cost of hiring the studio

s

£ 400

Number of students in the class

n

Fee for one student to attend the course

f

£

Profit made by the teacher

p

£ 160

$$p = fn - s$$

$$f = \frac{p+s}{n}$$

$$n = \frac{p+s}{f}$$

$$s = fn - p$$

The cost of hiring the studio

£ *s*

Number of students in the class

n

Fee for one student to attend the course

£ *f*

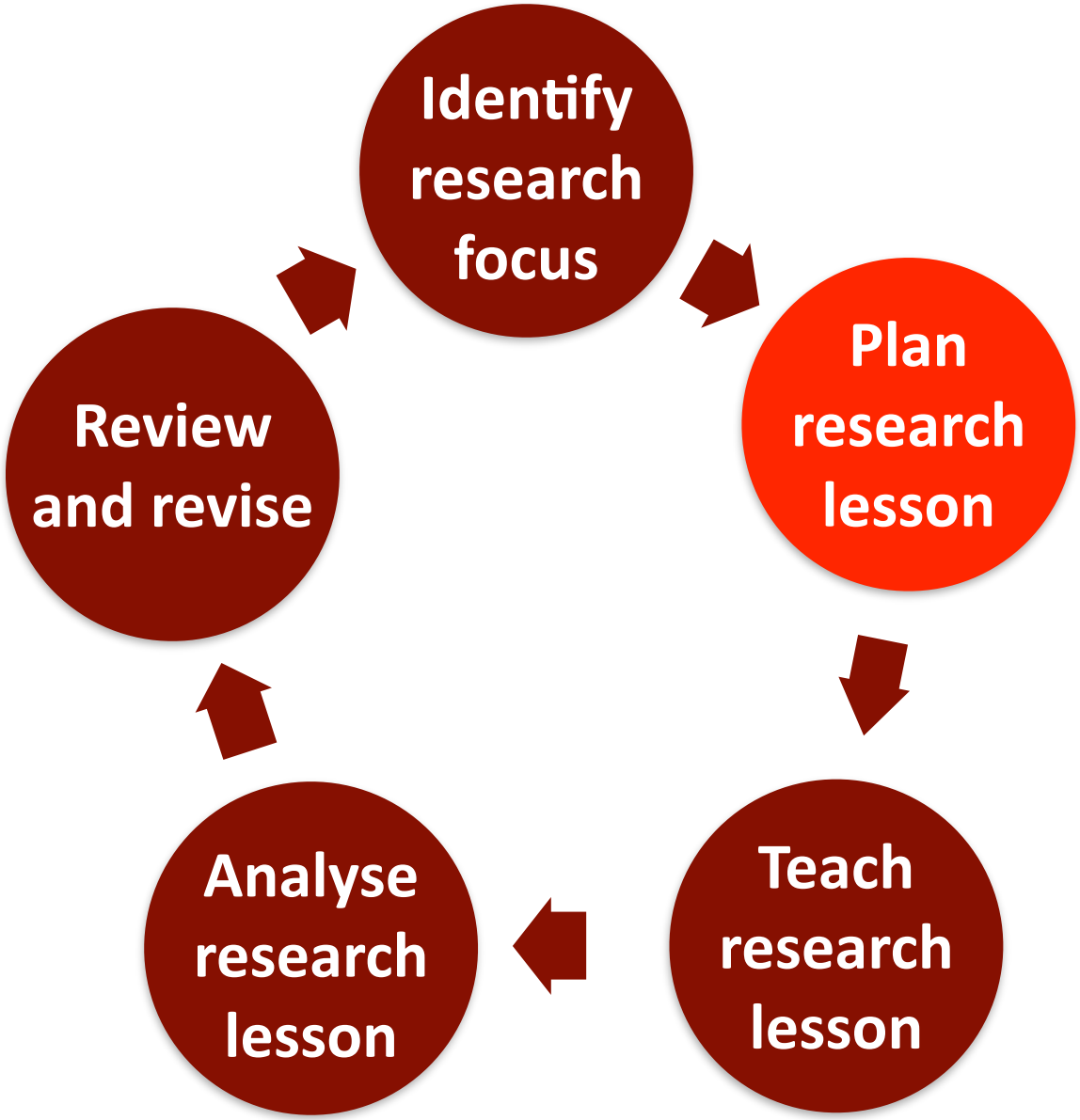
Profit made by the teacher

£ *p*

Research Question

How can we enable students to plan strategically and monitor their approaches more effectively?

Japanese Lesson Study Model



Teachers, collaboratively, plan:

- **The research objective**
- **The task tom be used**
- **The needs of particular students**
- **The place of the lesson in the sequence**
- **The phases within the lesson and their purpose**
- **The key questions that will be posed**
- **Anticipated student responses**
- **Responses to students' responses**
- **How 'success' may be recognised**

Phases of the lesson

Presentation (Hatsumon)

- Teacher presents the problem
- Students discuss the problem

Developing a solution (Kikan-shido)

- Students develop ideas individually
- Students share ideas
- Teacher observes students, makes notes for later

Comparing strategies (Neriage)

- Students share their solution ideas with whole class
- Students critique solutions, identifying strong and weak points.

Summarising and reflecting (Matome)

- Teacher summarises group findings, identifies important ideas, generalises
- Students summarise what they have learned themselves

Neriage

The ancient Japanese technique of layering, cutting and recombining different colours of clay, creating an intricate pattern.

"Japanese word for the whole class discussion phase of structured problem solving. It is the core of teaching through problem solving. This happens after students have shared various solution strategies. During this phase, students, carefully guided by the teacher, critically analyse, compare and contrast the shared ideas. They will consider issues like efficiency, generalisability, and similarity to previously learned ideas."

(Akihiko Takahashi)

Outbreak

A disease has started to spread around the city. If you get the disease you only have hours to live.

Our city has been put under quarantine; no one in or out.

The good news is you are able to help.

The scientists from the Research and Development Department have worked flat out and have managed to put together two vaccinations.

Outbreak

- Vaccination A is 100% effective and costs £12.00 per vaccine.
- Vaccination B is 70% effective and costs £5.20 per vaccine.
- We have a budget of £5,000,000

Your task is to recommend:

- How many of each vaccine should we make?
- Who will get those vaccines?

Outbreak

Occupation	Number in population
Medical workers (doctors, nurses)	75600
Key service workers (electricity, refuse)	113000
Food shop personnel	113000
Farmers and food producers	85100
Other shop workers	104000
Other professionals.... teachers, lawyers, etc.	123000
Other trades people ... decorators, plumbers, mechanics,	85100
Retired people	86400
Students and school students	94600
Children under 5	66200
Total	946000

Research lesson – lesson plan

Anticipating student responses

- In a preliminary lesson, the class attempt the task individually in silence.
- Responses are collected and analysed according to the approaches taken.
- Teachers prepare formative feedback questions for students.

Issues arising from initial attempts

Detailed calculations before planning:

- Start at the top of the list and calculate cost of vaccinating medical workers

Ignore constraints.

Do not justify decisions made.

Leap to conclusions:

- "Vaccine A is more effective so just use that"

Don't understand the concept of a budget

Overwhelmed by the large numbers

Don't grasp meaning of calculations

Don't understand "effectiveness" of each vaccination:

- "70% effective so 70% must survive".

Become confused between numbers representing money or people

Anticipated issues table

Key Issue	Suggested questions or prompts
Students start detailed calculations before planning an approach	• Describe in words a plan for tackling this problem. • What are the key decisions you have to make? • Which information are you going to focus on at the start, which will you ignore?
Students ignore one or more constraints.	• Do you have enough resources for your solution? • Have you made enough vaccine for everyone? • Have you wasted any money? • Have you wasted any vaccine?
Students do not justify decisions made.	• Why have you chosen to allocate the vaccines in this way? • How can you be sure this is the best solution?
Students leap to conclusions	• Have you taken all the issues into account? • Could you vaccinate more people if you used some of vaccine B? • Could you save more lives if you used more of vaccine A?

Key Issue	Suggested questions or prompts
Not understanding "budget"	• What is your main objective when trying to solve the problem? • Are there any more lives that you could possibly save?
Overwhelmed by large numbers.	• How much money do you have remaining in your budget? • How many more vaccines would you be able to purchase with this amount of money?
Not grasping meaning of calculations	What does this figure represent? Is it how much money is left over or how much money has been spent? Does it represent an amount of people?
Only writing numbers; no justifications	Where have these figures come from? Do you know what they represent? Are you able to justify why you have used these numbers?
Not understanding "effectiveness"	If 1000 people were given vaccination B, how many would be likely to survive?
Confused between numbers	• Can you think of a way of distinguishing between numbers that represent different values? • How can you distinguish between values that represent people or money?

	Strategic planning	Monitoring work
Little progress		
Some progress		
Substantial progress		
Task accomplished		

	Strategic planning	Monitoring work
Little progress	Carries out operations with figures but shows little strategic awareness that will lead to a solution.	Carries out calculations without stopping to reflect or think about what is being achieved or alternative approaches.
Some progress	Carries out appropriate and correct calculations but does not take constraints into account.	Considers alternative approaches by comparing own method with others, but this has no impact on own approach.
Substantial progress	Works towards a solution logically reaching a viable solution	Considers the work of others. Compares approaches and uses them. Finds it difficult to discriminate efficient/ inefficient approaches.
Task accomplished	Arrives at a solution having considered alternatives.	Engages thoughtfully with the work of others. Selects and uses powerful approaches.

	Strategic planning	Monitoring work
Little progress	Carries out operations with figures but shows little strategic awareness that will lead to a solution.	Carries out calculations without stopping to reflect or think about what is being achieved or alternative approaches.
Questions	*Can you write a plan for completing the task? What other information must you consider?*	*When you have finished this calculation, what will you do next?* *How will you organise your work?*
Some progress	Carries out appropriate and correct calculations but does not take constraints into account.	Considers alternative approaches by comparing own method with others, but this has no impact on own approach.
Questions	*Are there other pieces of information you have not thought about?*	*What ideas does your partner's work contain that may help?*
Substantial progress	Works towards a solution logically reaching a viable solution	Considers the work of others. Compares approaches and uses them. Finds it difficult to discriminate efficient/ inefficient approaches.
Questions	*Can you think of another method? What be the effect on the outcome?*	*Which idea is more powerful? Which method would work with different numbers?*
Task accomplished	Arrives at a solution having considered alternatives.	Engages thoughtfully with the work of others. Selects and uses powerful approaches.

Green sheet to encourage monitoring

Analysing Partner's work

1. Describe briefly what your partner has done.

2. Why do you think they have done this?

3. How is this different from your approach?

4. What impact will this have on your next attempt at the problem?

Using Sample Student Work

Medical workers are the most important they all get (A)

$75600 \times 12 = 907200$

~~12~~ $5000000 - 907200 = \boxed{4092800}$

Remaining budget

Farmers and Students are important for the future, they all get (A)

$(94600 + 85100) \times 12 = 2156400$

$4092800 - 2156400 = \boxed{1936400}$

Using Sample Student Work

All get (A) 946000 × 12 = 11352000

Over £5M buget

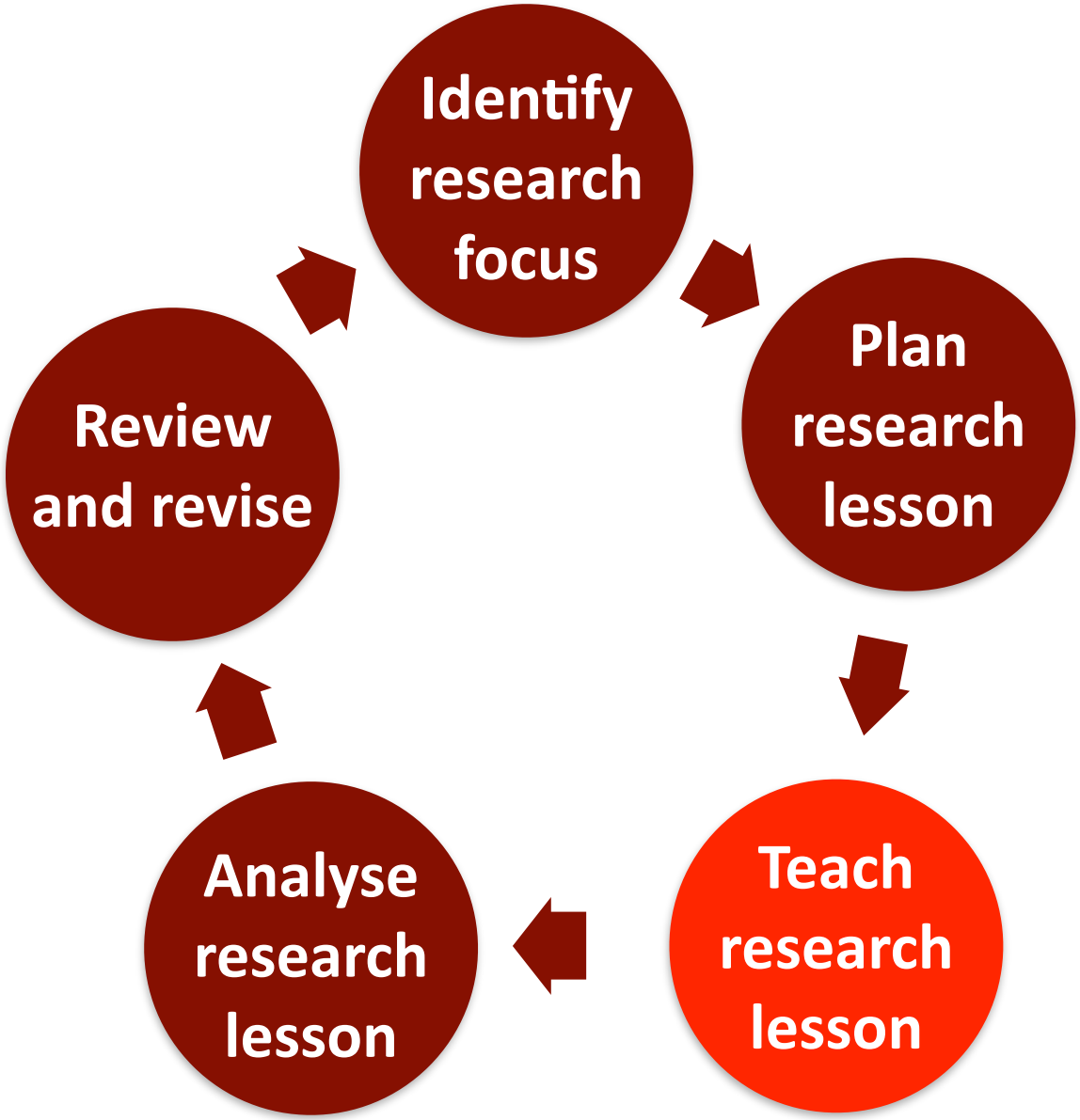
All get (B) 946000 × 5.2 = 4919200

Within £5M buget (80800)

80800 ÷ 12 = 6733 can have (A)

Teacher planning meeting

Japanese Lesson Study Model



Observe and describe:

Teaching:

- what are the most effective prompts and questions?

Learning:

- how do selected students respond mathematically?
- what do they discuss?
- how do they reason?

Mathematics:

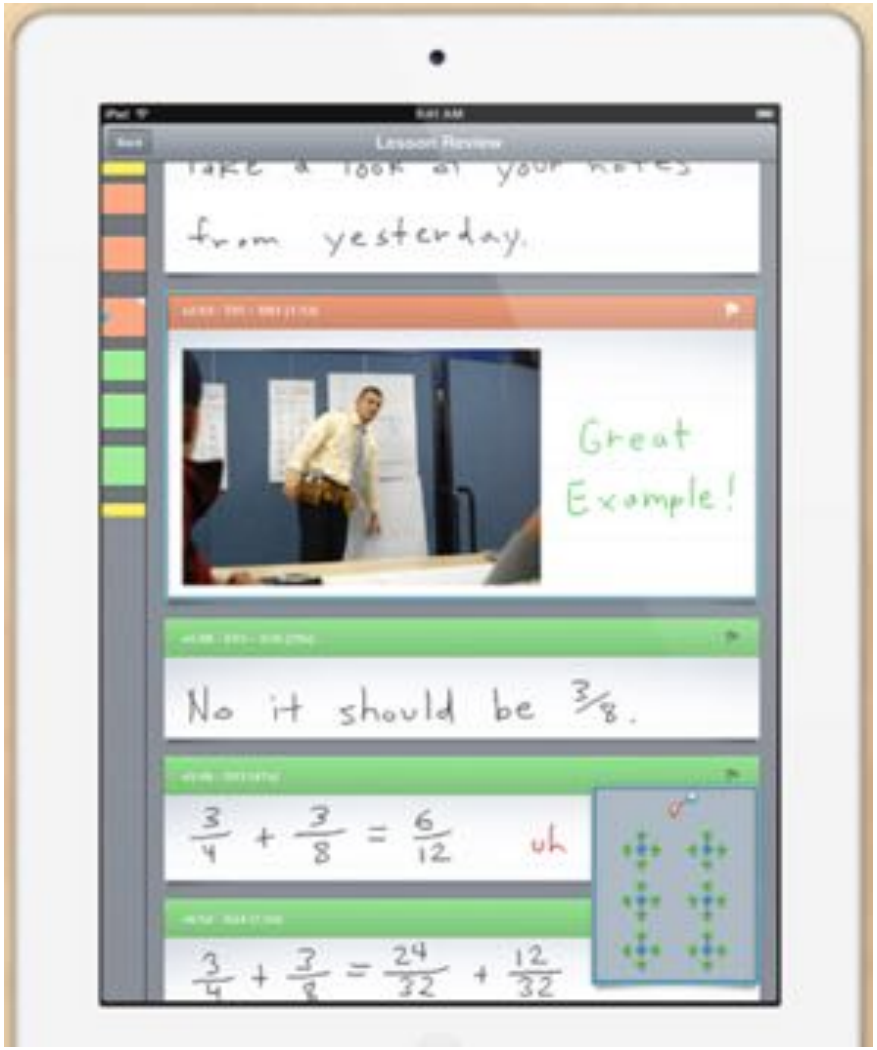
- how does the mathematics flow and develop during the lesson?

Observation of the lesson

The focus is on the learning not the teaching.

The lesson is a group product, and all members of the group are responsible for the outcome of their plan.

Tools such as the app "lesson note" may be used to record lessons.



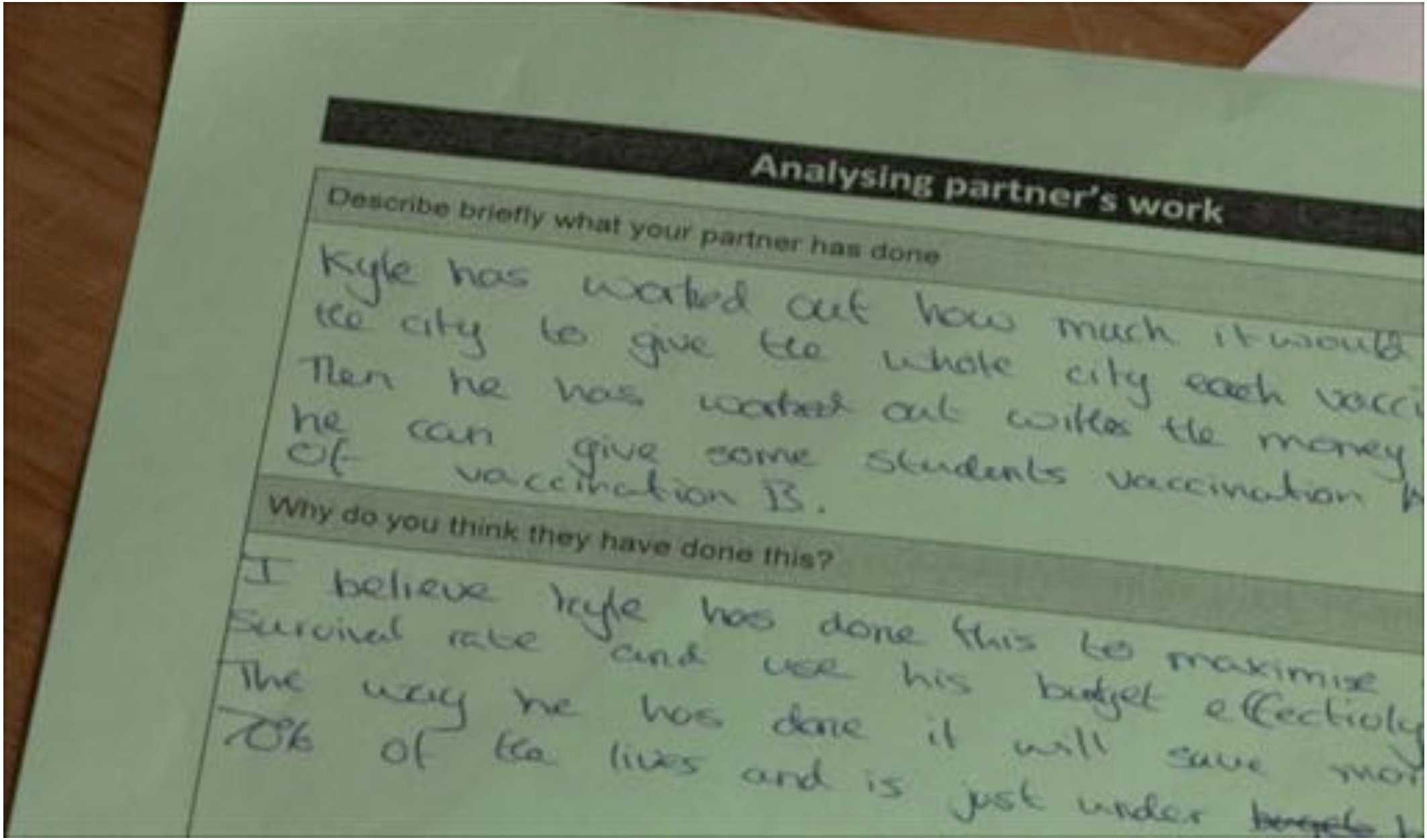
Lesson Outline as it happened:

- **Recall the task**
  " For 30 seconds just tell your partner what you remembered."
- **Review (In silence)**
  "In silence, re-read the task from the sheet".
- **Clarify the goal (In pairs)**
  "Think and discuss what it is you are trying to achieve.
  Write down this goal."
- **Interpret partner's work (In silence)**
  - Describe what your partner has done
  - Explain why they have done this.
  - How is their approach different to yours?
  - How is this going to have an impact on what you do next?
- **Explain partner's thinking to them (In pairs)**

Recall the task

1 min 13

Explain partner's thinking to them



Analysing partner's work

Describe briefly what your partner has done

Kyle has worked out how much it would
the city to give the whole city each vacci
Then he has worked out wither the money
he can give some students vaccination A
of vaccination B.

Why do you think they have done this?

I believe Kyle has done this to maximise
Survival rate and use his budget effectively
The way he has done it will save mor
70% of the lives and is just under

Produce joint solution
Review goals - still within budget?

60s

Whole class presentations

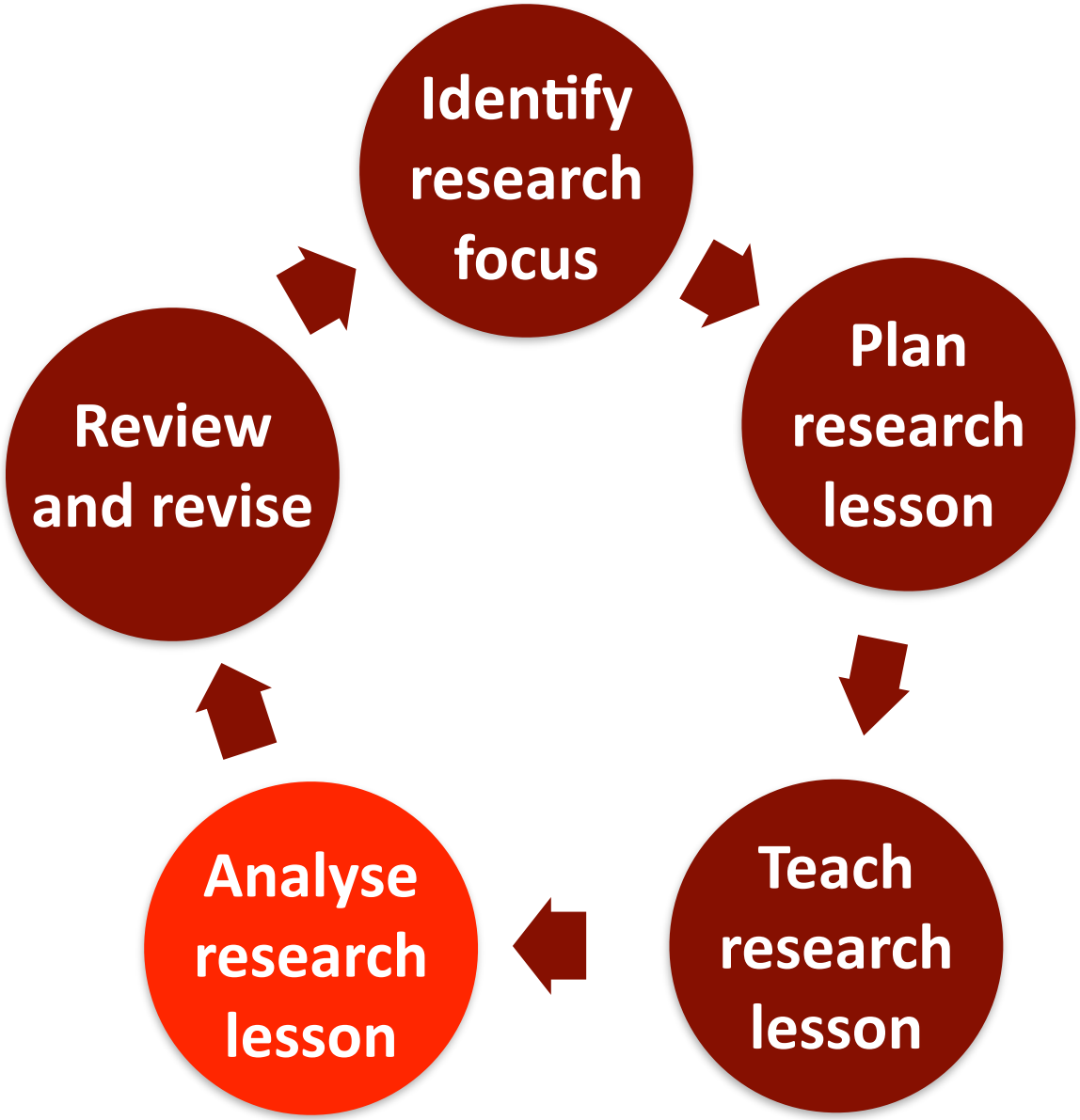
60s

Planning for next time

Summary of lesson structure

- Introduce and tackle task individually
- Recall task
- Review and feedback
- Clarify the goal
- Interpret partner's work - silent
- Explain partner's thinking
- Produce joint solution
- Review goals
- Whole class presentations
- Planning for next time

Japanese Lesson Study Model



- Same day debriefing includes a facilitator, the teacher, observers, a commentator (koshi).
- Teacher describes the lesson; reasons behind decisions made; departures from the plan.
- Observers describe what they saw in the target students. Discussion focuses on the research question.
- Commentator relates observations to research and discusses implications for future.

Planning fostered by interpreting each others' work

Post-lesson discussion

Context taken seriously

- Students discussed who would look after the children that were going to be saved.

Planning was fostered by interpreting each others' work

- Making students interpret and explain each others' approaches was effective.

Student discussion didn't always lead to collaborative work

- One pair discussed each others' work but then continued individually

Students misunderstanding own calculations

- Subtracting money from population!

Misconceptions

- Do we give vaccinations after getting disease?
- Giving two vaccination B's. Does this make them 140% effective?

"Moral v Mathematical" approaches

- Deciding who will live and die first or after deciding how many vaccines are available?
- How will we decide which 50% of farmers to save?

Class presentations

- Do students adopt approaches because the teacher has chosen them?

Choosing the presentations; Maths, Morals and Monitoring

Contribution by the “Koshi”

Issues arising: Planning and monitoring

Stopping students and asking them to explain and plan:

- “Pens down and tell each other what you are doing.”

Teacher’s questioning encouraged students to think strategically:

- “How do you know when the money will run out?”;
- “Are we achieving our goal?”
- “Could you do better than vaccinate everyone with B?”

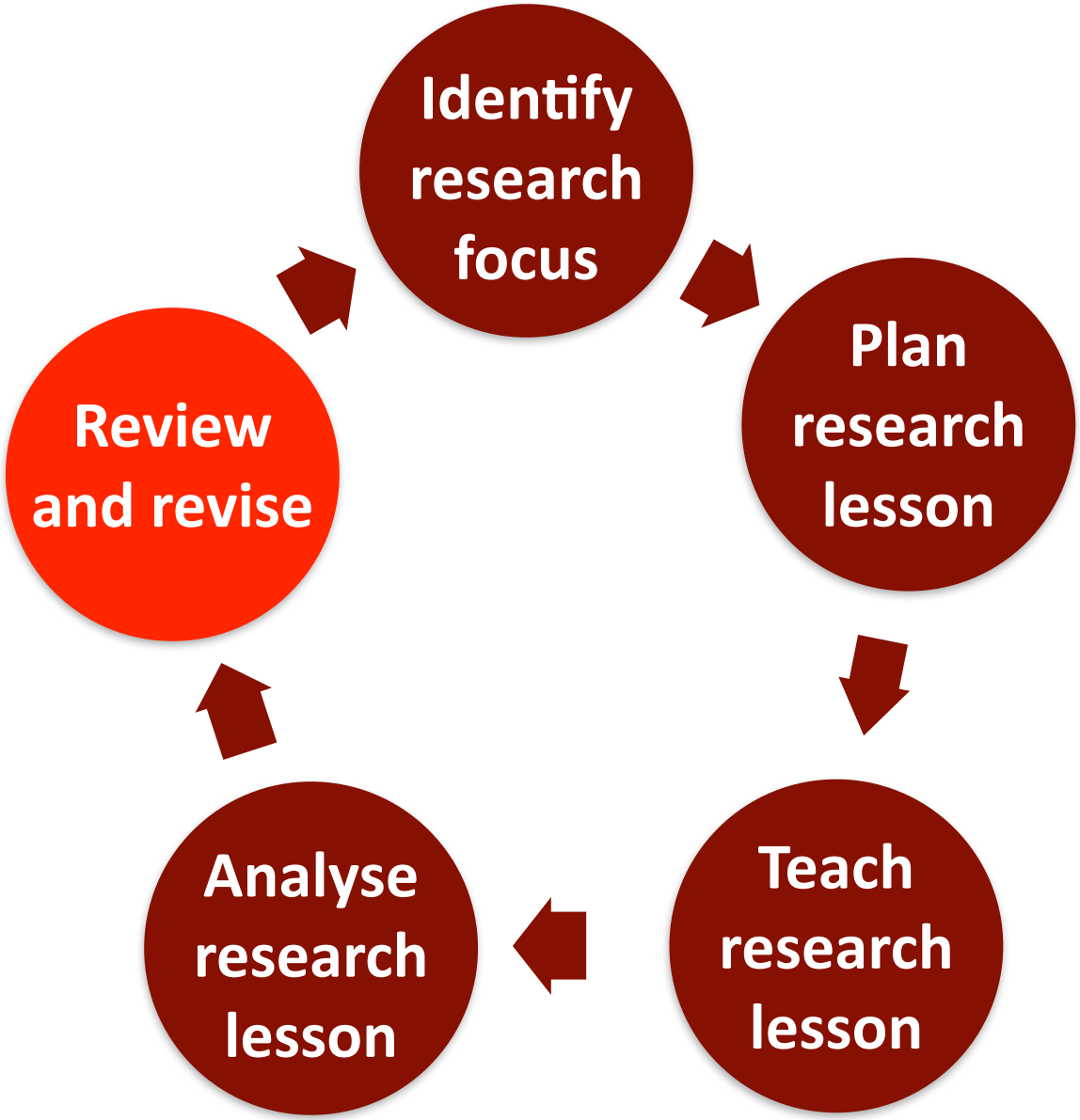
Redrafting.

- Redrafting helped students to re-think their approach.
  “Put down the steps you were taking.”

Stopping class and considering students’ work

- Carefully chosen examples focused on strategy.

Japanese Lesson Study Model



- Teachers review the lesson objectives.
- Review each phase of the lesson, the flow, the timings, the prompts.
- Revise the tasks and questions.
- Revise the anticipated student responses - using actual responses.
- Uses sample student work to illustrate success criteria.
- The lesson may be re-taught.

What has been learned so far?

Teaching problem-solving processes

- Interpreting processes and recognising progression is difficult
- Representing is often confused with communicating
- Analysis needs to be linked to metacognition and controlling the direction of the problem solving

Process v content debate

- Initially, teachers taught techniques just before the problem
- Too many processes addressed in some lessons
- Students unaware of process goals

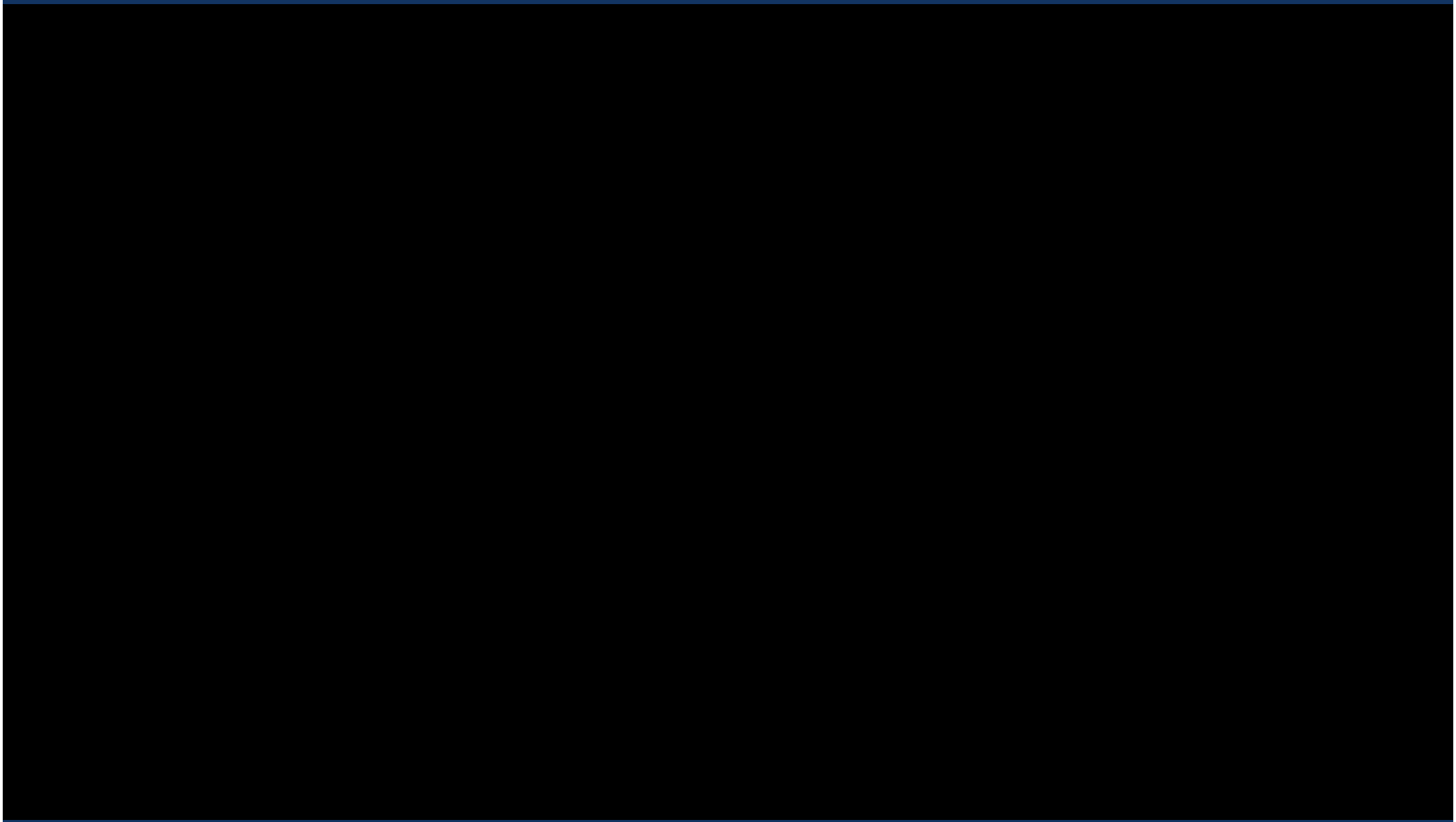
What has been learned so far?

Choice and adaptation of tasks

- Details matter; using student-generated data can make it more relevant, but can also trivialise it!

Structure and pace

- All lessons overestimated what students could do in time.
- Approaches that were designed to focus students' attention had the reverse effect.
- Students need more time to engage with the reasoning of others, particularly in their presentations.
- Give time for refinement. Posters encourage this.
- Neriage was not understood.
  Convergence on agreed method is probably unlikely to happen.
- Need to focus more on detail of processes.



The toolkit

A toolkit to support the needs of all the lesson study group.

LeMaPS: Lessons for Mathematical Problem Solving

Search

Home
- About the project
  - Project aims
  - Project Background
  - Problem Solving
  - Lesson study communities
  - Partnership, Activity and Timeline
  - Participants
  - Downloads from workshops
- Toolkit
- Diary pages

Toolkit

Welcome to the toolkit main page

This toolkit contains a number of pages of explanatory material about Lesson Study.
As we further develop this, we will attempt to add downloads in the form of presentations, videos, documents and activities of various kinds.
It is structured around the following questions:

Why is Lesson Study for Problem Solving needed?

- Lesson Study for Professional Development
- Lesson Study for Problem Solving
- Lesson Study for enhancing the curriculum

How can I create and sustain a lesson study group?

- Organising a lesson study community

How can I plan for a research lesson?

1. Overview
2. Planning Meeting 1
   - Identifying the research focus
   - Selecting a suitable problem solving task
3. Planning Meeting 2
   - Working on the task and anticipating student responses
   - Recognising progress
4. Planning Meeting 3
   - Completing the lesson plan

What happens on the day of the research lesson?

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For further details go to

http://www.nottingham.ac.uk/education/research/crme/index.aspx

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