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First Steps in Initial Teacher Training: Encouraging Reflection on Children's Learning in Mathematics

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Some years ago, following an Association of Teachers of Mathematics General Council Meeting, Laurinda Brown asked what we did with mathematics postgraduates in our first sessions of a teacher training course. The question was intriguing, but was difficult for me to answer in the context of a complicated undergraduate modular BEd programme. Following Circular 9/92, in partnership with the Gloucestershire Association of Secondary Headteachers, the college instigated a one year PGCE course in September 1993. The course is an innovative one involving full partnership with schools in Gloucestershire and the surrounding area (see Bloomfield et al, 1994). In relation to this article, the relevant significant features of the course are the focus on how children learn and the relatively early move towards supported placement in secondary schools. In this more focused context, Laurinda's question made much more sense. Our answer to it was outlined at the May 1996 AMET / BSRLM Conference in Loughborough, where others were asked to think about and to compare their first sessions. This is an outline of our first steps at Cheltenham & Gloucester College of Higher Education (CGCHE) and the issues raised in discussion with colleagues from other institutions.

Underpinning Practice with Theoretical Structure

The place of theory within teacher training has been discussed by Johnson (1995). She recognized the varying practice that occurs within seemingly similar courses of initial teacher education. Sinkinson (1996) has commented on the need for mentors to touch on the theoretical aspects of teaching and learning. How then do we help postgraduates and their mentors to make meanings from theory and practice?

Our understanding of adult based learning is based on *conjectural model* that is introduced in the college in September and has the elements of the following theoretical perspectives:

- Social constructivism. Learning is dependent upon context and content and is essentially social and cultural with a key role for the expert:novice relationship. Context is not simply imposed on people- it is something which to a large extent
- Siegler's model of domain-specific and domain-general knowledge (1989). This presents the view that we build up a repertoire of specific strategies for resolving specific problems. Different strategies may develop for different domains. In Initial Teacher Education (ITE) strategies are context bound, e.g. to the immediate 'domain' of each lesson or to the specificity of the subject matter.
- Learners' perception of reality will determine their thinking. One model for this is G. H. Mead's 'symbolic interactionism'. A more recent one is Ros Driver's 'alternative frameworks', from the domain of science. (Clough & Driver 1986)

- Accepting that we can function at different levels of competence simultaneously. We may be incompetent to teach, yet competent to think intelligently about teaching. We can therefore keep both thinking about teaching and doing teaching as interwoven developing strands while on a PGCE course.

In relation to this fourth perspective, Munby & Tochon (1993) talk of a wave function between these two aspects of thinking and doing. This model of learning can be usefully applied to our work with children as well as with adults, and is a 'specific learning' model (Rogoff, 1984). It postulates that we develop cognitive skills and other skills in relation to a particular task - as opposed to following a 'central processor' model (thinking as a unitary process). Clearly both models have their validity, but the 'specific learning' model is at least as important as the 'central processor' idea and is part of an analysis of adult learning based on a multiplicity of different ways of learning. A major component must be the context i.e. the school: learning to teach by teaching, by watching teachers, by talking to teachers, by telling your peers about what you have discovered (presentation to peers of first assignment). These varied modes (watching, talking, telling, teaching, writing) are synthesized through the 'teacher as researcher' methodology outlined below.

Teacher as Researcher

In order to provide practical structures for organizing one's own behaviour, perceptions and beliefs, we offer the 'teacher as researcher' methodology. This contains simple data collection and data analysis techniques, which invite contrasts between, for example, ethnographic and systematic observation approaches. These approaches make it possible for postgraduates to record their own thought processes and activities in terms of explaining, predicting and experimenting. They are predicated on the belief that the reflective practitioner can only develop by using these structures that provide practical opportunities for thinking about their own professional development. It also provides a framework for professional conduct, incorporating some case study approaches (of individual pupils and of whole-school issues) and some action research approaches (of one's own teaching).

The Model of Initial Teacher Education

The model of ITE employed is based on the following beliefs:

- that any or all of the above factors may influence the teacher mentors and that they may learn as significantly while mentoring as the trainees being mentored
- that a core curriculum for ITE must seek to permeate all levels of ITE within a school based partnership; the main mechanisms for this permeation are the writing, production and dissemination of both general and subject specific material, together with regular meetings of appropriate groups of teachers
- that partnership involves full consultation about the content and delivery of the ITE curriculum

This model of ITE, which is based on active partnership between schools and HEI, has implications for the kind of support materials which are used within the course in terms of:

- Medium - materials produced jointly and disseminated to all people and institutions

involved, workbooks with activities, which include video.

- Contents - materials contain a core of knowledge and shared understanding of the needs of postgraduate teachers and of staff in schools.

Sequencing - like Bruner's spiral curriculum, so that, broadly speaking, without being too prescriptive, those issues considered to be fundamental are dealt with on several occasions in increasingly complex ways.

The First Sessions in Mathematics Education

The first general professional studies unit (Professional Preparation) therefore contains a number of chapters on educational theory (Scott Baumann, 1995). The focus on theory is we seek to help postgraduates' meaning-making by working in school, albeit within a somewhat unrealistic classroom! The intention is to place 'theory', generated by Piaget, classroom, to see theory in practice and to look at models of learning. The first step is courses make the decision to begin with the postgraduates' own experience of earning mathematics. The CGCHE mathematics education programme begins with a discussion of the nature of mathematics - what we would hope it would bring to the education of the child and what are the aims for the subject in secondary school. Of necessity, of course, we discuss the structure of the National Curriculum in mathematics. Our initial focus is on attainment target Ma1 'Using and Applying Mathematics'. How can postgraduates start to make sense of this attainment target which is often in conflict with their own experience of learning mathematics.

The approach adopted within the Gloucestershire partnership is to start with children in a classroom. Postgraduates go to a local 11-16 school and take over the timetable for 2 hours for a group of 60 Y8 pupils. In 1995-96, the school used was Cirencester Kingshill School. Integral to this process is the model of ITE which sees the professionals involved as partners in the testing and development of educational theory in practice. In this situation Dave Eacott, the Head of Mathematics and Kieron Smith, the mathematics postgraduate there, helped to plan and introduce the session.

The task

The task for the pupil group was an investigative activity that the year group would have undertaken normally as part of their programme of study. The worksheet forming the starting point for their work is based on an investigation of the number of lines needed to connect various numbers of people. It is a familiar problem which we introduced, using toy telephones made out of plastic coffee cups and string, by Dave and Alan. Pupils were encouraged to record their mathematics by talking to their group and then to write down what they had said ("Say it first, then write it down"). Groups of pupils then worked on the problem and extension activities where appropriate for two hours, producing a display of their work for the end of the session. The postgraduates had to plan for this phase of the work, with, on average, two postgraduates working with seven or eight school students. Classroom management as a whole was in the hands of the experienced teachers. With at times up to twenty plus adults around and sixty pupils this was not a major problem. In short this was an unrealistic scenario and deliberately so. The intention was to focus on pupils' learning in a very supportive environment for the postgraduates.

Reflection on video

Martin (1994) has described the difficulties many students face when having to record their

reflection in writing. In order to encourage reflection about what was happening in the moment and record it for more deliberative reflection at a later stage, we adopted the following strategy. Dave Eacott and Alan Bloomfield were available to foster the postgraduates' understanding of new knowledge in action and the possible meanings of events. Postgraduates had the chance to step back from groups and to 'see' pupils learning mathematics at firsthand, together with the challenge of noticing and putting into words what they had seen. By taking a postgraduate aside and getting them to talk to camera, we recorded immediately what they had seen as significant. We recorded on video both what happened and the postgraduate's 'noticings'. At the end of the two hours after the pupils had left, the postgraduates recorded their final reflections on what they had observed during the session. This use of video is consistent with their experience in college during the first two weeks of the course, when they watch video footage of Gloucestershire teachers talking about their lessons, as well as seeing excerpts from the lessons themselves. In the Professional Preparation lectures and workshops, the postgraduates use video footage to analyse different teaching and learning styles. In fact several extracts from this mathematics video taken at Cirencester Kingshill were later incorporated into the second Professional Preparation video and accompanying workbook for the second half of the Autumn Term.

Categorising Postgraduate Observations

Postgraduates were working with groups of pupils chosen from across the range of attainment within the school. Their comments then were categorised as shown in the examples below.

Pupil motivation and interest

There seemed to be a lot of interest straight away. Pupils went straight into working individually rather than waiting for others to take the initiative.

Time went quickly, even though a two hour session can appear quite daunting.

Ways of seeing

I was impressed how they would come up with ideas in a different way to what we expected... For instance one pupil was trying it on his calculator and he found the formula by trying numbers ..

I was surprised how quickly some of them visualised it straight away. They got an image of what was happening with the telephone lines before they actually looked for patterns in the numbers. It (the rule or formula) was much more obvious to them from the diagram they were using than from the table of data. It came more from the diagram by counting the number of people and the direct lines from each person than from the chart. They could follow it on one by one from the numbers in the table but they couldn't jump (to a rule or formula).

Encouraging explanations by school students

One group picked it up quicker than the other. We encouraged that group to explain it to the other group. They quite enjoyed that and it clarified their understanding just by explaining. They had the idea but for them to put it into words, they found that quite difficult, but it actually clarified their thoughts.

One child immediately came up with the answer about 5 to 10 seconds after the start. I asked him to explain to everyone else, just to make sure he understood what he meant.

Range of understanding

There was one boy in our group. He just said it's N times $(N-1)$ over 2. He got it so fast and then wanted to get onto something else.

We took then away into the corner of the room (to do the problem practically). With two people, add another person ... add two lines. With three people, add another person ... add three lines. It was really difficult for them. In about five metres (going back to the tables) from there to there, two of them had forgotten completely.

Looking Back Two Terms Later

This event took place at the start of the course. A group of postgraduates were interviewed at the start of the Summer Term to find out their views of what had happened two terms earlier. What did they remember and what was valuable about the experience? The effect of the day appears different in some ways from our original intentions. The focus for the postgraduates was on being a teacher, but the range and variety of children's thinking is still an important aspect of what was remembered.

Being a teacher

It was the first contact that we had in a teaching maths environment. I found it quite exciting; it gave me a buzz. It was good to see how the children worked together in a small group, (containing) children of different abilities.

In our group one of them was messing about and I had my first chance to tell somebody off. The first thing he said was "I'm not telling you my name".

It was like the first time it really did hit me how it was going to be different being a teacher to being a pupil in the classroom ... how the tables were turned from my being told off to my having to try and motivate pupils to buckle down to what they were supposed to be doing.

Even though it was really small groups if it had been any bigger I would have been petrified ... Knowing that it was a smaller group and you had someone else there with you, then it was okay to go in and start being a teacher. If I hadn't had that experience and going into a classroom and somebody calling me "Miss" I'd probably have fainted. You were being looked at as a teacher.

Pupil motivation and interest

How easy it was to get them on task. I must have had some sort of thoughts prior to going there that it was going to be quite difficult and I was quite surprised because it was almost quite an exciting situation for the children because they had all these teachers coming in. giving them two teachers per group. Because they were actually quite excited it took very little prodding to get them on task.

There was one girl for me. I always thought I'm going to go into teaching. I've always been interested in maths I'm going to be the best maths teacher and everyone in my class is

going to be interested in maths. There was this one girl the first time I've ever taught in a group of six and one girl just wasn't interested ... I tried and I tried to get her interested. She had ability as well, a lot of ability, but she wouldn't try. It was so frustrating.

Range of understanding

(I noticed that) even within our small group of eight, the difference in the abilities of all of them. I didn't expect that and it really did prepare me for when I went into school, where you've got classes of 30 with that sort of ability range.

That's one thing you learn. You don't assume too much knowledge and you're never surprised at how much or how little they know. A specific child, who you had to explain to physically, move him about. When he got back to his seat, he hadn't a clue what he'd been doing. I'll remember that for the rest of my life!

Ways of seeing

It was well organised and there was a lot of maths in the actual investigation, which made it slightly easier. After about five minutes one of the lads that was in our group got it straight away and said "This is easy" and started to mess about. We got him to extend it to different amounts of lines and different people wanting to talk to each other. I'm not sure whether he understood the algebra but I think he understood what the problem was ... I think he visualised it more than anything. He just saw how it was solved straight off and I think if you've given him something completely different with the same answer he wouldn't have been able to get it. He just found that particular problem easy to grasp.

It was good to see the different ways in which pupils tackled questions: some went for the formula straight away, others could visualise it, others drew a table and others had to physically get telephone lines. It was good trying to get to their level. That was always going to be an eye opener throughout my teaching career.

They said "But I can do it this way" I said "How many have I got and how many have you got". She said "You've got one more than me ... you've done it wrong." I thought you go in, be a teacher, they listen to you and take on board what you've said and try and do it, but no ... kids are like that. They will do it the way they want to do it! Definitely.

The session at AMET: May 1996

The session at Loughborough provided an opportunity to answer Laurinda's question and to allow others to talk about their own approaches. At the end of the session we discussed the Gloucestershire approach and focused on the early move into schools and the use of video. Comparisons with what happened at the start of other courses were made and questions raised about the validity of the exercise. After an initial brief discussion of what happened early on in other courses, we looked at excerpts from the video. There was then a discussion of the strategies used within the early stages of the CGCHE programme. The main themes of that discussion and people's comments are recorded below. At this stage of the discussion colleagues were not aware of the postgraduates' more recent recollections of the event that are summarised in the earlier section of this article.

Classroom management issues

One of the things we were looking at was whether this exercise addresses students' initial concerns on a PGCE Course (mainly about classroom management and control) or whether it leap-frogs them ... whether it says; 'that's not something we are interested in ... far more fundamental is this issue of noticing'. We wondered whether it would be better to get students to address their concerns, then to say to them when they will be addressed during the course.

Perhaps the significant thing for the students was that they haven't had their concerns made explicit but they could still be thinking about classroom management issues, while the exercise is nominally about noticing pupils' learning.

Looking at learning

We agree that it's very important to look at learning. We look at learning by getting students to look at their own learning first and that had quite a few problems associated with it, as well as quite a few advantages. But you're looking specifically very early on, the very first things you do, at children's learning in the moment when they are actually doing some mathematics.

We were surprised at the speed with which you set up an activity and get the students into school to carry out the activity with children. Generally we were saying we spend more time in those very early stages setting up a group identity and in getting students to reflect on their own learning especially in mathematics and how that has been different at various times in their lives before we begin to look at children's learning.

The focus is on the students' own mathematics and on building group dynamics within the group ... We were wondering about that 'groupness' that we talked about earlier, because on watching the video it seemed the students were very willing to talk. It was interesting that that might come from the very nature of the task that you set them, because it was very unthreatening, focusing on the children rather than focusing on what the student-teachers had been doing.

Links with general professional studies

There seemed to be an expectation that students should adopt the language used in their reading ... the way you put the questions was linked to the Bruner section. We try to work by getting the students to articulate what they see ... I introduce the technical language to

help them to articulate what they see and not to impose a structure. The issue is about what comes first.

The significance of course structure

It's going to make a big difference how soon the students are going into school as to what sort of order you do things in ... I'm not sure what the relationship is exactly, but I think it's important.

Different groups of students each year led to very different things ... what happens in those first few weeks is so coloured by their existing beliefs and attitudes. I've set up some one to one work between students and children focusing on what children know about a specific topic ... that seems to have been more successful in getting students to start interacting with children, what children know and how to talk to them.

In our structure, they have just come off two weeks of Primary school experience, so a lot of what you (CGCHE) are doing (in the visit to school), when the group dynamics are settled, has to be mapped into our first session!

Discussion

Many of the decisions we make about our individual programmes for initial teacher education are heavily influenced by factors outside the control of mathematics educators in Higher Education Institutions. There are external pressures such as the shift towards partnership with schools, the time spent in schools and the requirement for profiling teacher competencies, as well as internal pressures, such as the aims of the partnership team from schools and college in planning the course. However there are still decisions left open to us that are the subject of differing conclusions - and hence debate. The discussion reported in this article has indicated how even the first stages of a mathematics education programme can differ significantly between different courses. There are fundamental decisions to be made about the ordering of reflection and awareness of theoretical frameworks. A major issue is the relationship between models of subject structure and general pedagogical models within Initial Teacher Education. It can be argued that each subject area has a distinctive culture of philosophy and beliefs about learning that differentiates it from all others both in 'substance' (the organisation of concepts & principles to incorporate the facts) and in 'syntax' (the set of ways in which truth, falsehood, validity & invalidity are established) (Schwab 1978). It is possible also that the thought processes of the learner may sometimes provide a barrier: the specific learning model may preclude generalisation from a professional preparation workshop to a mathematics lesson, unless we can continue to make connections.

At the AMET session, colleagues from other institutions were offered a description and analysis of what we do with postgraduates in training. By bringing in evidence of what we do in schools and within the college, we provided an insight into our two subcultures and how they are able to work in partnership. This allowed all of us involved at AMET to gain an awareness of the decisions we have made and of our own reasons for making them. Closer partnership with schools has offered us the opportunity to start with children's learning in the classroom. However it is unlikely - and probably undesirable - given the very different situations within our institutions and partnerships, that there can be total agreement on the most effective way to begin a course.

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Problem-solving: Still a Problem in Primary School

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This article is a report and evaluation of the design and use of an exemplar problem-solving package with teachers in Bradford first schools. It reveals that primary teachers continue to have difficulty in integrating problem-solving activities into their regular mathematics work with children and were generally not able to adapt an exemplar problem-solving package for their own use.

Problem-Solving Not Integrated

Since the publication of the Cockcroft report in 1982, primary school teachers have been exhorted to include problem-solving and investigational work in their mathematical teaching styles.

Since Cockcroft I have been involved, first as a primary mathematics consultant and then as a teacher trainer, engaged in both initial training and INSET, in encouraging and helping teachers to implement these teaching styles.

Initially examples of problem-solving and investigational work tended to be one-off examples offering the teachers the experience of these teaching styles. The objective was always to integrate these new teaching styles into the normal classroom work, into theme work and cross-curricular work. Indeed, one of the expected advantages of these approaches was that children would meet the same mathematical content within different contexts and therefore be given the opportunity to transfer acquired knowledge from one situation to another.

Problem-solving and investigational work was given statutory recognition in 1989 with the arrival of the National Curriculum with two of the fourteen attainment targets being Using and Applying Mathematics. This was later revised to become one of five sections of the programme of study.

Part of my work involved observing students on placement, with the result that I visited a large number of primary schools. By September 1995, some thirteen years after the Cockcroft report and six years after the introduction of the National Curriculum, my main observation on children's problem-solving work in school was that it still consisted mainly of one-off problems. Most teachers had not managed to integrate this teaching style into their normal classroom work. This failing was not just apparent to the outsider observer: teachers themselves felt there was a problem.

In-Service Support

My college provided in-service training to local schools on any topic they asked for - thus providing a ready means to keep a finger on the pulse of teachers' concerns about mathematics education. The majority of requests for INSET at that time were for the 'Using and Applying Mathematics' section of the National Curriculum.

On my introductory visits to the schools that had requested sessions on 'Using and Applying Mathematics' I came across the same story time after time. Teachers were genuinely convinced of the need to do problem-solving and they understood the reasoning underpinning its inclusion in the National Curriculum. Reasons they gave for its importance included "so that they can use maths in real problems" and "so that they understand maths is useful". However, they felt they needed help with the implementation of problem-solving. It was my aim to provide a means of helping them.

At this time, the first schools I was involved with delivered much of their curriculum through a series of themes,' carefully chosen to cover the areas of the National Curriculum. Common themes were such as: ourselves, the neighbourhood, minibests, and the seaside. I decided to design an exemplar problem-solving pack giving some examples of how mathematical content could be learnt through the cross-curricular approach used for the rest of the children's learning. The local schools all had claris works on Apple Mac computers. I thought that if I produced a pack of problems on the computer, as a sort of word processing template, I would be able to advise teachers on how to alter the problems to fit their particular theme.

The Exemplar Problem-Solving Package

I chose the theme of houses as my example, for two reasons: (a) it is a common theme in primary schools and so the pack could be used without alteration; (b) it is similar in structure to many other primary school themes.

My intention was to use the package with different schools who had asked for INSET on 'Using and Applying Mathematics', to evaluate its effectiveness and to make the necessary alterations to improve it. The evaluation was to consist of participant observation of the Inset sessions, observation of its use in school and discussions with teachers on their use of the material.

The pack consisted of an introduction and a set of activity pages.

Introduction

This provided problem-solving activities based around the topic of houses. These activities were developed loosely around the mathematical content areas of sorting (including permutations and combinations), number bonds, patterns/multiples, repeating patterns and particular situations. There were also suggestions for activities using measurement and data-handling.

Each content area had an introductory page giving advice for choosing situations. For example, the number-bonds introductory page included this:

Choose a situation where pictures can be drawn or models can be made where colours can be used to partition the set in a variety of ways e.g. tulips in window boxes.

Choose a situation where a picture or a model can be made modelling a partitioned set where numbers of small objects can be shared in a variety of ways between the partitions of the set, e.g. children upstairs and downstairs in a house.

Next the objectives of the activity were given in terms of the National Curriculum, followed by some guiding questions for the teacher:

- 1) Do the children understand the problem?
- 2) Can they find some solutions?
- 3) Can they record their solutions in some way?
- 4) Can they sort their solutions in a helpful way?
- 5) Can they find all the solutions?
- 6) Can they explain that they have all the solutions and why?
- 7) What strategies do they use - practical, moving things around, or mental?
- 8) Can they invent a problem of their own?

Activity pages

These were written for a teacher or adult to read to the children, but could be left on the table as a reminder. Each activity page had the objectives and questions from the introductory page on the back so that they could be used by the adult not familiar with the pack. For example:

Children in my house

(Picture of a house with an upstairs and downstairs supplied)

There are _ children playing in my house. They can be upstairs or downstairs.

How many different ways can they be in the house? Are you sure you have found all the different ways? What about if there were a different number of children?

Some of the children are girls and some are boys: what can you do about that?

Working with the teachers I discovered that for some of the questions they needed solutions themselves. This might seem surprising to some readers, but it reflects the continuing, well-documented insecurity of many primary school teachers with any slightly unfamiliar mathematical tasks. For example, with an activity called House-sorts: [a house can have a window, a house can have a door, a house can have a chimney, or a mixture of these. How many houses can you make? How do you know you have got them all? Can you invent your own house problem? (Garages, floors, trees, etc.)], it was necessary to provide the following guide to a solution:

	chimney	chimney	not chimney	not chimney
	window	not window	window	not window
door	1	1	1	1
not door	1	1	1	1

Using the pack in INSET

In the first session we discussed Using and Applying Mathematics, the role of context in mathematics learning and ways in which teachers could develop mathematical contexts (much of the introduction to the pack). I then introduced the pack and explained how it could be used.

In the second session teachers came prepared with themes they were going to use in the next half term. The themes they chose included: seaside, artistic achievement, people who help us, minibeasts, and clothing. The teachers brainstormed these themes using the house pack as a basis.

Results with a theme like 'Seaside' included a number of contexts for sorting activities

(permutations and combinations), such as lollies with stripes, ice creams with different flavoured balls, sand castles with towers, flags, or shells, different shaped and patterned flags. Here I needed to provide some advice about choosing combinatorial situations - for example, the teachers chose problems for young children involving many flavours of ice cream and the solutions became rather difficult to work out.

Activities on number bonds that emerged within this theme included: Ice cream sundaes (eating balls of ice cream), fish in nets, and children in boats.

I identified the need to provide some activities within the content area of 'money'. We developed the idea of a seaside shop from which the children could decide what to buy. Then I took the results of the brainstorming away, wordprocessed them, (often simply by altering the pages in the housepack) and returned them to the schools on disc and on paper for use with their children. The schools were also given a version of the house-pack on disc so that they could do their own alterations on a subsequent occasion for a different theme.

Session 3 of the INSET varied from school to school, but always involved some form of evaluation of the project.

The Reality- the results

The teachers enjoyed the brainstorming and found it helpful, they all used some of the activities with the children, indeed they were very pleased with the results of the activities and many were displayed on the wall around the schools.

The most-motivated of all the teachers rewrote all the activity cards in her own words on small cards (her normal mode of delivery for such work with children). Nobody used the computer to generate a new problem-solving pack. A variety of reasons were given for this, such as: the computer did not work, the printer did not work, lack of confidence knowledge of the computer. None of the schools has used the problem-solving pack to help them design mathematical problem-solving for other themes, as I intended.

My own evaluation

I was disappointed. I had thought that my exemplar problem-solving pack might really have been helpful and that it would only require minor alterations to make it user-friendly, enabling teachers to generate their own material quickly and easily.

Where did I go wrong? Perhaps because I enjoy using the computer to generate written material, I may have over-stressed its role. It is clear now that it was a mistake to put such an emphasis on its use. I mistakenly thought that teachers would find it easy to adapt the activity pages that I had designed. They did not, possibly because they were not confident about using the computer, but more likely because they simply did not have the appropriate time with the computer - and if they did they had other priorities. I had not taken into account when and where many teachers do much of their preparation for school - in the evening and in their living rooms and kitchens, not in front of a computer. The technology for many is still card, careful printing and sticky backed plastic.

If I am able to continue this project, I would continue to use a similar pack (albeit revised in the light of this experience), but I would concentrate on developing the brainstorming, with the teachers producing rough notes of ideas which could be delivered to children verbally. I would drop the emphasis on the computer, for teachers do not need to spend their life

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doing what they might perceive to be unnecessary word-processing.

However, at the moment the tide has turned again, with schools no longer requesting INSET on Using and Applying Mathematics, but help with basic numeracy. I wonder why.

Student-Teachers' Conceptions of Mathematical Proof

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The successful teaching of mathematical proof depends crucially on the subject knowledge of mathematics teachers. Yet the knowledge that teachers have of mathematics has become a matter of major concern in both pre-service and in-service teacher education. While this debate has largely focused on primary teachers, much less has been said about the subject knowledge of secondary mathematics teachers. This paper reports on an initial analysis of a small-scale investigation into trainee secondary mathematics teachers' conceptions of mathematical proof. Some tentative implications are drawn from this preliminary analysis. For example, it may be that while the least well-qualified trainee secondary teachers may have the poorest grasp of mathematical proof, the most highly qualified may not necessarily have the specific kind of subject matter knowledge needed for the most effective teaching. The methodology of concept-maps used in this study may provide a valuable approach to gathering insights into students' understanding of mathematical proof.

Concerns about the Teaching of 'Proof'

A number of concerns have recently emerged about the teaching of mathematical proof at school level. The London Mathematical Society, for instance, suggests that "most students entering higher education no longer understand that mathematics is a precise discipline in which "logical exposition and proof play essential roles." (LMS, 1995, p. 8) Among the possible causes they infer that the UK National Curriculum for mathematics may be distorting the notion of mathematical proof (*ibid*, p. 25). In a similar vein, the Dearing review of qualifications for 16-19 year olds expresses concerns about the "limited perceptions of the role of proof" amongst A-level candidates and recommends that the mandatory core at A-level should be reviewed (Dearing, 1996, p. 96-98). Such a review has taken place and the new core for A-level mathematics does indeed contain a greater emphasis on mathematical proof.

The LMS report also notes that "to improve what is taught and how it is taught, we must raise the competence and confidence of those who choose to become mathematics teachers." (*ibid*, p. 22) A crucial factor in this is the knowledge that new entrants into mathematics teaching have of mathematics. In this paper I focus on one pertinent aspect of the mathematical subject knowledge of pre-service secondary mathematics teachers, specifically their conceptions of mathematical proof. I present a preliminary analysis of data from a small-scale investigation which suggests that whilst the least well qualified trainee secondary teachers may have the poorest grasp of mathematical proof, the most highly qualified may not have the specific kind of subject matter knowledge needed for the most effective teaching. This accords with the current focus on the subject knowledge base of mathematics teachers.

The Subject Knowledge Base of Mathematics Teaching

The subject knowledge that teachers have as a basis for their teaching has become a matter of particular concern in both pre-service and in-service teacher education. This is especially so in the case of the teaching of mathematics in primary schools. Ofsted, for instance, claim that whilst teachers' command of mathematics is adequate in 90% of

schools at Key Stage 1 and 75% of schools at Key Stage 2, in only 10% of primary schools is it good or very good. (Ofsted, 1995, p. 9) Research by Aubrey, amongst others, has provided more detail on aspects of the influence of primary teachers' mathematics subject knowledge on how they teach (see, for example, Aubrey, 1996). As a result of this concern about primary teachers' knowledge of mathematics there have been a number of initiatives to support teacher development. These include the provision of short courses, supported by publications such as Haylock (1995). More recently, the Teacher Training Agency has seen fit to impose an "Initial Teacher Training National Curriculum for Primary Mathematics", which specifies the essential mathematics subject knowledge that must be taught to all trainee primary teachers (TTA, 1997a).

In contrast, the situation is often viewed as being somewhat different for secondary teachers of mathematics. After all, the argument goes, secondary mathematics teachers are, in general, mathematics specialists and so subject knowledge ought to be secure. Indeed, Ofsted suggest that 60% of mathematics teachers have a good or very good command of their subject and infer that anything less than adequate is due to schools using some non-specialist teachers at Key Stage 3 (Ofsted, 1995, p. 10). Yet it is worth investigating further whether all is well with the subject knowledge of secondary mathematics teachers. Certainly, evidence from a study of biology and geography teachers suggests that, for teachers of these subjects, both subject knowledge and pedagogical knowledge can be unsatisfactory (Hoz *et al*, 1990). What is more, this particular study found that the gaining of experience does not necessarily improve this knowledge, although the teachers in the study did appear to master subject knowledge better than they did pedagogical knowledge. There is some similar evidence about secondary mathematics teachers. In a detailed study of prospective mathematics teachers' knowledge of functions, Evan, for instance, found that these student teachers did not have a complete conception of this important part of mathematics (Evan, 1993). For example, appreciation of the arbitrary nature of functions was missing, and very few could explain the importance and origin of the univalence requirement. This limited conception of function appeared to influence the student teachers' pedagogical thinking.

Given that the majority of UK secondary mathematics teachers enter the profession by completing a one-year postgraduate course of initial teacher education (the PGCE), the basis for their subject knowledge is, in the main, developed during their specialist undergraduate course. Indeed, government requirements state that the content of a PGCE entrant's previous education must provide the necessary foundation for work as a mathematics teacher. During the one-year initial teacher education course, the emphasis has to be on transforming sound subject knowledge into secure pedagogical content knowledge (Ruthven, 1993). Consequently, teacher educators need to be confident that student teachers on a PGCE course have sound content knowledge, particularly with respect to essential components of mathematics, such as mathematical proof.

The Role of Proof in Mathematics and Mathematics Teaching

Mathematical proof is an essential component of mathematics and is arguably what distinguishes mathematics from other disciplines. As such it should be a key component in mathematics education. Yet providing a mathematics curriculum that makes proof accessible to school students appears to be difficult. Proving, it seems, either appears as an obscure ritual or it disappears in a series of innocuous classroom tasks in which students learn to spot patterns, but not much else (Hewitt, 1994). For example, Schoenfeld (1989), reports that even when students can reproduce a formally-taught Euclidean proof, a significant proportion conjecture a solution to the corresponding

geometrical construction problem that "flatly violates the results they have just proven." On the other hand, when the chosen proof-contexts are data-driven, with students expected to form generalised conjectures and search for counter examples, Coe and Ruthven (1994) find that students' proof strategies are primarily empirical. It seems that the generation of numerical data becomes the object of the exercise and any notion of deductive argument is rejected. Balacheff (1988, p. 222) similarly reports the occurrence of what he refers to as "naive empiricism."

A likely relevant issue is that proofs are often thought of solely as standardised linear deductive presentations. Indeed, this is how proofs are frequently presented. The form of two-column proofs taught in a number of countries entirely fits such a model. Proof can, however, take a number of forms. Balacheff (1988, p. 216), for instance, contrasts what he calls pragmatic and conceptual proof. >From a different perspective, Leron (1985) talks about "direct" and "indirect" proofs. A further distinction suggested by Hanna (1989, in Hanna and Jahnke, 1996) is that there are proofs that prove (and do no more) and proofs that explain. This latter form of proof, Hanna suggests, demonstrates not only that a statement is true, but also why it is true.

While underlining the central importance of mathematical proof, the above considerations say something about the difficulties pupils have in learning what constitutes a proof and indicate some ways in which proof might be taught in a more meaningful way. Of course, there is a model of progression in mathematical reasoning embedded in the UK National Curriculum for mathematics (DFE, 1995). This indicates that, in the earliest years, pupils can be taught to recognize simple patterns and make predictions about them, ask questions such as *what would happen if?*, and understand simple general statements such as *all even numbers divide by 2*. Following this, pupils are expected to make conjectures, make and test generalisations, and appreciate the difference between mathematical explanation and experimental evidence. Only the older, more able, pupils in the 15-16 year age range are expected to extend their mathematical reasoning into understanding and using more rigorous argument, leading to notions of proof.

All this suggests that the teaching of mathematical proof places significant demands on both the subject knowledge and pedagogical knowledge of secondary mathematics teachers. Yet to be awarded Qualified Teacher Status in the UK, intending secondary mathematics teacher have to demonstrate, amongst other things, that they have a secure knowledge and understanding of the concepts and skills in mathematics at a standard equivalent to degree level to enable them to teach mathematics "confidently and accurately." (TTA, 1997b) The above discussion of mathematical proof suggests that intending secondary mathematics teachers need to have an extremely secure subject knowledge base of mathematical proof if they are to teach it accurately and with confidence.

In the next section of this paper I describe some aspects of a small-scale study designed to illuminate the conceptions of mathematical proof held by trainee secondary teachers of mathematics. The aim of the study is to provide evidence of how secure mathematics student teachers are in their conception of mathematical proof. The methodological tool chosen to reveal the student conceptions is based on the idea of the concept map advocated by Novak and Gowin (1984), principally as an aid to meaningful learning. More recently, concept mapping has been suggested both as a tool for assisting the teacher to teach and the learner to learn, and as a research and evaluation tool (Markham *et al*, 1994). Because this study is very small-scale, clearly any conclusions must be very tentative. However, readers may find the

methodology described an interesting approach for gaining insights into students' understanding of the nature of proof.

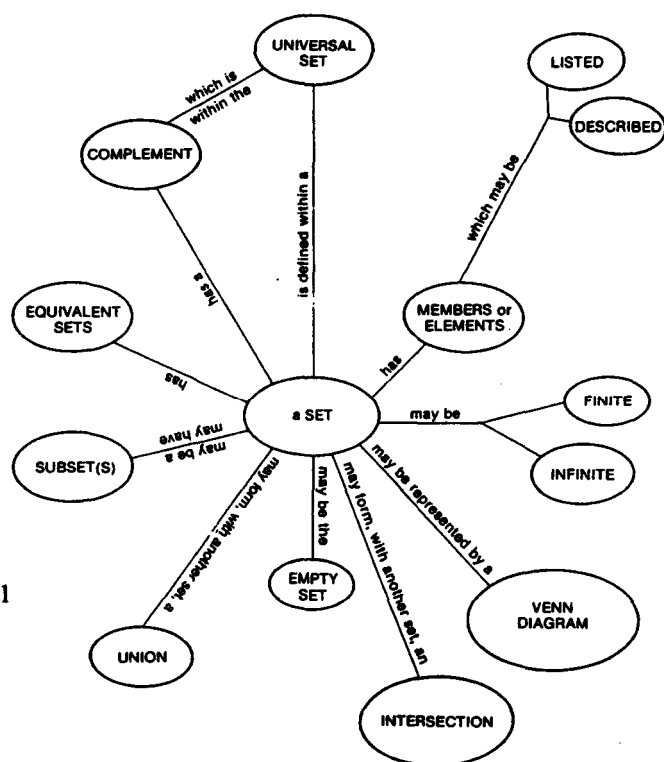


Figure 1

Concepts Maps of Student-Teacher Knowledge of proof

A concept map is, at its simplest, a graphical representation of domain material generated by the learner, in which nodes are used to represent domain key concepts and links between them denote the relationships between these concepts. In this way, a concept map provides an explicit representation of knowledge. An example of a concept map focusing on elementary set theory (see Figure 1) is provided by Orton (1992, p. 169).

Characteristic elements to note are the nodes, which represent key ideas, and the linking lines which express the relationship between these ideas.

The theoretical foundation of concept mapping is Ausubel's theory of learning, which suggests that meaningful learning depends on integrating new information into a cognitive structure laid down during previous learning. The argument put forward by Novak and Gowin, amongst others, is that concept mapping resembles the cognitive structure developed during learning. It appears that neurologists tend to agree with this proposition.

To generate a map of their conception of mathematical proof, student teachers followed a version of the suggested method for producing concept maps. Step one is to produce a list of key terms that the students associate with mathematical proof through a group brainstorming session. The following is the list that one group of 25 students generated (in no particular order):

Euclidean
trial and improvement
axioms
definitive

logic
graphical
syllogism
lemma

explanation
precision
observation
theorem
irrefutable
postulate
hypothesis
proposition

examples
reasoning
general case
assumptions
deduction
by contradiction
implies
abstraction

Step two is for each student teacher, individually, to produce their own representation of their conception of mathematical proof using any or all of the above key terms (or others that they might choose), arranging them on a blank piece of paper from their own perspective, and joining key terms in what they consider a meaningful way for them, using lines and words indicating relationships between the key terms they use.

Below are examples from three student teachers, all of whom have degrees in mathematics but with varying classifications. The maps were drawn during week 16 of a 36-week course. These particular examples are selected to illustrate the range of responses from the trainee teachers to the task of drawing up a concept map and to test the method of analysis. Further analysis of data from a larger sample of trainee teachers is needed before any firm conclusions could be drawn either about the quality of the subject knowledge of trainee mathematics teachers or of the validity and reliability of the concept map method. Nevertheless, the data given below does raise some interesting questions. Figure 2 is produced by student-teacher A, whose degree classification is a pass.

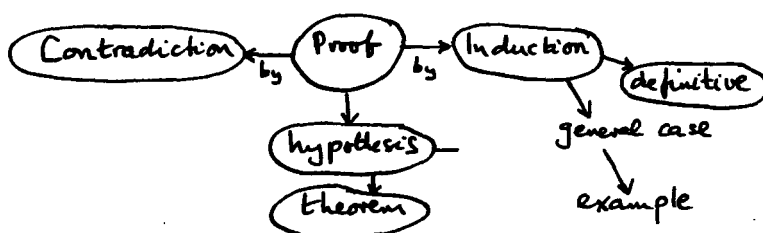


Figure 2

Figure 3 is produced by student-teacher B who has a third class honours.

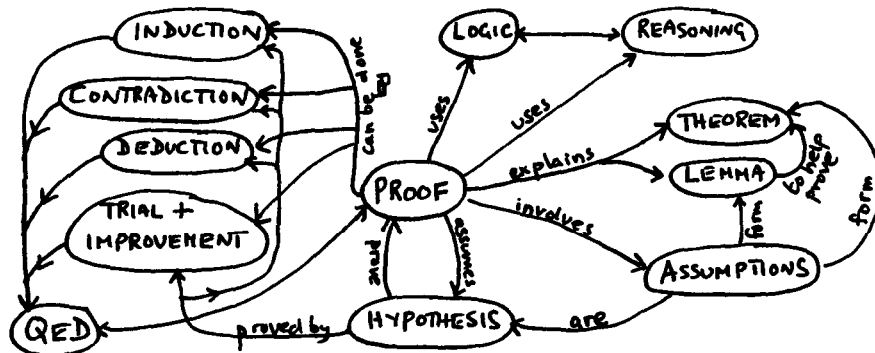


Figure 3

Figure 4 is produced by a student-teacher C with an upper second class honours degree.

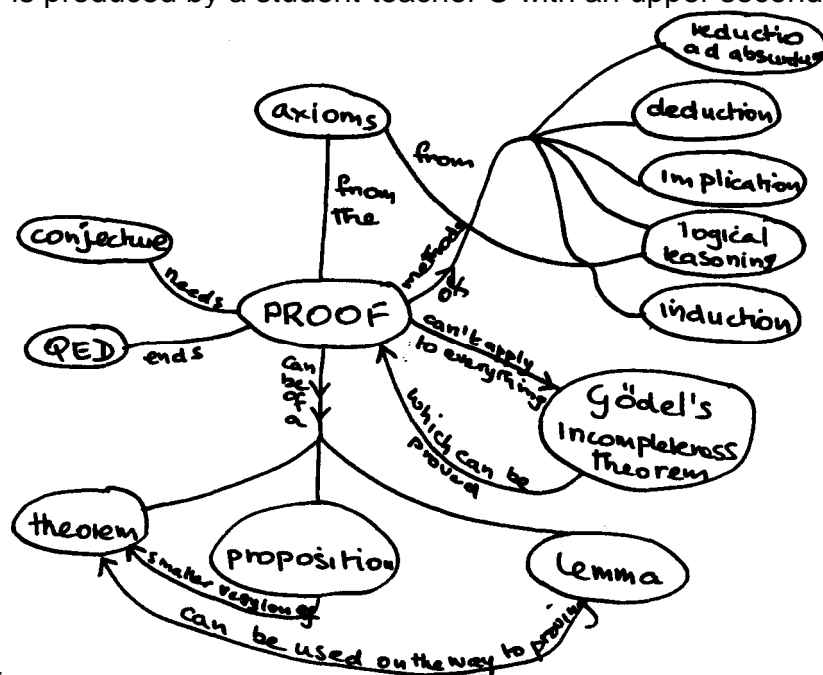


Figure 4

Before embarking on any analysis of these concept maps, it is important to recognise that cognitive structures and concept mappings are highly personal, as each individual's knowledge is unique. Concept maps are inevitably idiosyncratic. There is no one "correct" concept map. However, this does not mean that all concept maps are correct. It is possible, for instance, to examine the key terms used and the way in which relationships between these key terms are specified. It may also be possible to identify errors, such as the absence of essential concepts or inappropriate relationships between concepts.

Analysis and Discussion

In this section I present a preliminary analysis of the above three concept maps based on two main criteria:

- the use of key terms: how many key terms are used and which ones are included
- the specified relationship between key terms: how many relationships are specified, how they are specified, and whether cross-links or multiple relationships are indicated

Such an analysis of the three concept maps given above is shown in Table 1. The first column indicates that the better the qualification, the more key terms the student uses. There is also evidence from the maps themselves that the most highly qualified student teacher produces what could be considered a more sophisticated map by introducing additional terms not in the list given above.

	Number of key terms	Number of relationships	Number of relationships specified	Number of crosslinks specified
Student A (pass)	6	7	2	0
Student B (third)	12	25	18	12
Student C (2:1)	13	16	16	3

Table 1

Nevertheless, despite using more key terms, the most mathematically-able student-teacher does not specify either the greatest number of relationships between key terms

nor the greatest number of cross-links between key terms. It is student B who does that. Such a preliminary analysis only allows some very tentative conclusions to be drawn but these may fit in with some other findings. One implication from the analysis might be that trainee mathematics teachers with the barest minimum qualification of a pass degree in mathematics might need considerable support in developing a secure knowledge base of mathematics. Given the largely school-based nature of initial teacher education this may be rather difficult to provide.

On the other hand, the evidence here might suggest that having the best qualification does not necessarily mean that the student teacher will make the most effective mathematics teacher. For example, while the student-teacher C has, arguably, the most sophisticated knowledge of mathematical proof, the concept map of student-teacher B might be considered somewhat the richer as it has more relationships between the key terms present. This fits with the findings of a study by the US National Centre for Research on Teacher Learning (NCRTL 1993) that "majoring in an academic subject in college does not guarantee that teachers have the specific kind of subject matter knowledge needed for teaching."

Given that the concept maps provided above were produced during week 16 of a 36-week course, it is possible that the concept map of student-teacher B has developed some of the linkages that reflect the transformation of subject knowledge into pedagogical content knowledge that is the aim of the initial teacher education course. Consequently, student-teacher B may well be the most effective and successful of the three teachers chosen for this analysis.

On the other hand, some of the differences between the concept maps of students B and C may be due to differences in their undergraduate courses. This possibility is supported by some of the findings of the NCRTL report, which is highly critical of undergraduate University courses - such as many in mathematics that require students to memorise massive amounts of information while paying little attention to the meaning or significance of the material covered. The suggestion is that once students graduate, they often think about mathematics as lengthy lists of facts with little or no consideration given to relationships among principles and concepts. This has the effect of making the transformation to effective pedagogical content knowledge all the more difficult. The NCRTL researchers did find a university-based course that seemed to make a difference. This course required students to reason about the subject, to argue about alternative explanations for what they encounter, and to test their ideas and those of others. Such academic interaction, the study found, tended to improve students' understanding of important concepts in the subject matter and, along with that, their ability to explain concepts.

Given the central importance of proof in mathematics and in mathematics education, the development of successful and confident secondary mathematics teachers depends both on sound subject knowledge built up at undergraduate level, and secure pedagogical knowledge developed during postgraduate initial teacher education courses. This demands that attention is paid both to undergraduate courses in mathematics as well as to courses in initial teacher education.

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Assimilation Versus Accommodation: A Piagetian View of Difficulties Encountered in Mathematics by Undergraduate Students

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This paper arises from a small-scale research study. Four undergraduates encountering difficulties with a variety of new processes in mathematics were observed to return to familiar methods. Three students seemed unable to accommodate to the new techniques, whilst the fourth used her knowledge of familiar methods to deepen her understanding of the new processes. New, more-complex notation is evident in all four cases and it is suggested that changes in notation may be indicative of a need for higher levels of cognitive reasoning and therefore may signal potential areas of difficulty for students. Whilst the cognitive processes resulting in accommodation require further and more-substantial research, it is suggested here that students seem more likely to accommodate new techniques if they recognise that they are required, can link them with previous methods, and have time to grow accustomed to new notation.

Introduction

I believe that a better understanding of the difficulties encountered by mathematics students would be obtained if such difficulties could be seen in relation to the students' cognitive skills, as well as in terms of mathematical content. This raises the questions of what cognitive skills are required to understand mathematics, whether students already have these skills, or if need to develop them during their course.

The cognitive development of children has been the focus of many investigations and the subject of a number of theories. Arguably, the most influential has been Piaget's theoretical framework, which proposes that the structure of a child's understanding of the world passes through an invariant sequence of well defined stages as he/she matures. The stages are determined by the expanding range of cognitive tools which the child uses 'to understand'- from physical experiences to mental images, thought experiments, and finally abstract formal reasoning.

Piaget's theory seems particularly relevant to mathematics because it is based on the development of logico-mathematical skills. His work explains and exemplifies the mismatches which can arise between adult intentions and children's understanding. It therefore provides guidelines for mathematics appropriate to a child's stage of cognitive maturity. Critical debate of Piaget's methods resulted in practical, child-centred approaches which encouraged cognitive development and, when transferred to the classroom, transformed educational practices.

This is just the sort of analysis which might benefit undergraduate mathematics, but unfortunately Piaget's work was focused on children, not young adults. He had very little to say about the formal operational stage, except that it is final because the reasoning structure is complete. Unlike previous stage descriptions, Piaget offers no further elaboration or analysis of the cognitive operations which develop, but ~ and Machado point out that:

although critics always refer to a single formal stage, Piaget himself reported the existence

of different levels of formal thought in his formal stage (e.g. early formal thinking and full operational formal reasoning).
(Lourenço and Machado, 1996)

There have been a number of proposals for analysing thinking in or beyond the formal stage. Arlin (1975) identified the formal operational stage with problem-solving and suggested that problem finding constituted a fifth stage; Labouvie-Vief, in considering lifespan development, suggests that cognitive change beyond formal operations demands:

a realisation that logic is merely a necessary condition and becomes a sufficient element of adult life only if subordinated to a hierarchically higher goal: social system maintenance. It is important, then to distinguish between logic as a goal and logic as a tool....
(Labouvie- Vief, 1980)

Both Goldin (1987) and Kaput (1987) focus on the understanding and appropriate use of representational systems used in formal reasoning. Commons, Richards and Kuhn (1982) regard Piaget's stages as a hierarchy of cognitive representations and their associated operations. They propose the higher order operations of 'systematic and metasystematic reasoning' as levels of thinking beyond the formal stage.

However, unlike Piaget's, none of these theories make a direct link between levels of reasoning and student difficulties when learning mathematics. A more practical approach centred on the learning process is required. Vygotsky focused on the current understanding of the child and suggested that new work had to be within a 'zone of proximal development' (ZPD) for learning to be successful. This has similarities with Piaget's model of how understanding is achieved at any stage of cognitive development.

Piaget suggests that a child constructs internal representations (schemata) of the objects, actions and feelings that they experience. New experiences are either *assimilated* to existing schemata, or schemata are modified (or new ones formed) to *accommodate* radically-different experiences. This adaptation allows the child to maintain a continually-changing state of cognitive equilibration; and the state of equilibration determines the stage of cognitive development.

I decided to use this model as a starting point for investigating the mathematical understanding of undergraduates. Piaget wrote:

Equilibration in my vocabulary is not an exact and automatic balance, as it would be in Gestalt theory; I define equilibration principally as a compensation for an external disturbance.
(Piaget, 1962)

The concept of equilibration therefore implies the resolution of tension. Students who encounter difficulties may fail to adapt sufficiently to accommodate new mathematical experiences but still need to maintain the process of equilibration. This study was therefore based on the hypothesis that:

students who encounter difficulty with new approaches to familiar areas of work will be drawn to assimilate new problems to familiar schemata instead of modifying existing schemata to accommodate the new approaches.

Observation was focused on students who were having difficulty in using new mathematical techniques to solve familiar problems.

The Classroom Study

Context

I was already assisting with a Foundation module for the first year of a two-year mathematics PGCFJBA with QTS courses¹. In addition, a colleague running a Linear Algebra course for the second year of a four-year BA with QTS² agreed to my presence as an observer. I had worked with both sets of students on previous occasions so they were familiar with my presence and willing to take part in the study.

The small numbers of students taking the courses (19 and 9 respectively) allowed me to observe the whole class and also to move unobtrusively to students encountering difficulties with the mathematics.

Data

Ericsson and Simon (1980) define three dimensions for verbal reporting during such mental activities:

- **talk/think aloud:** subjects verbalise thoughts as they occur; undirected probing may be employed but information reported is determined by the subject's focus of attention
- **concurrent probing:** subjects are instructed to report on specific aspects of the activity as they occur; this requires additional processing by the subject and results in information of interest to the researcher
- **retrospective probing:** subjects may be prompted to recall the activity by the use of general and/or specific questions; information may result from cognitive activities, such as inference or rationalisation, occurring after the activity

Factors affecting the validity of verbal data include:

- environmental influences, including the presence of the researcher
- incompleteness: both lack of probing and directed probing can result in the omission of relevant thought processes
- subjectivity: the researcher may influence the data obtained, and its interpretation

Additionally, verbal data which is inconsistent with observed behaviour or idiosyncratic could still be a valid source of information within this study.

¹ These courses for intending mathematics teachers take non-mathematics graduates with an element of mathematics in their degree and mature students with mathematical experience, respectively.

² A mathematics and education degree with qualified teacher status.

I was concerned to obtain accounts of student's thinking at the time of difficulty, rather than a retrospective recall that might be incomplete or rationalised. Previous experience led me to expect that these students would probably, without prompting, discuss difficulties with each other, with their tutors, and with me. I therefore decided to use my established role both to provide effective support (or a student's work, and to encourage verbalisation of the student's thinking - a form of 'think aloud' with additional directed probing.

Audio-visual recording of complete verbal protocols and associated written work seemed likely to interfere with this informal approach, whilst taking notes could give rise to incompleteness and subjectivity. Directed probing was already open to the second criticism, so I made notes at the time, including some verbatim reporting, and wrote up observations in more detail later.

Observation notes were also taken for the whole of the Linear Algebra session. This gave a more complete view of the experience of students observed encountering difficulties by placing problems in a broader context. It also provided a record of the progress of another student as he successfully negotiated some of the mathematics which others found difficult.

Observations

Students A and B were observed during a Foundation course session when they were consolidating their understanding of A-level mathematics. Students C, D and E were observed during a linear algebra class when work on non-absorbing Markov chains was revisited, and then absorbing Markov chains were introduced.

1. Student A was revising differentiation and having difficulty following an example in a textbook which converted an equation to its parametric equivalent before differentiating. He had returned to the non-parametric version and been unable to differentiate by this method. Student A could not recall seeing parametric equations before, so notation in terms of a third variable was totally unfamiliar and not related to his experience of differentiation. He was familiar with differentiation of non-parametric equations. A brief explanation using a different problem enabled the student to follow the textbook example. However he remained unconvinced that the parametric approach was necessary. He returned to his familiar method and experienced the problems which the parametric method avoided. We discussed the value of understanding parametric equations as an end in itself. The student had focused on solving the problem and seemed surprised that it provided an opportunity to learn more mathematics.

2. Student B was revising geometry and getting to know Geometer's Sketchpad. He had written a macro (a short program) to draw three circles each centred on a different corner of any triangle once one corner had been specified. In practice the number of circles drawn varied from none to all three. I observed student B working through the possible starting specifications (e.g. side a, angle B, side c) and noting what happened. When asked why he was using this approach he said that he had previously worked with computers and knew that he would be able to find out which specifications worked 'properly' by checking all possible combinations. He agreed that this would not explain why they worked. I suggested that taking a geometrical view might be helpful. Student B checked through the macro, found no obvious geometrical mistakes and went back to his combinatorial approach. When he had checked out all the possibilities he found that the order in which sides were specified resulted in different numbers of circles for the same specified corner. He seemed to realise that there was some form of convention operating, but did not relate it to geometrical conventions such as the direction in which angles are

measured. When this was suggested he acknowledged the possibility but continued to use an empirical approach rather than proposing and testing a hypothesis based on geometry. He saw no reason to consider unfamiliar geometrical notation when he could use his computer experience.

3. Student C was using a graphic calculator to obtain eigenvalues and then checking her answers by trying to remember and follow the method through by hand. When asked why she was doing this she replied: "It's still quite fresh, this stuff; we still want to know what we're doing." Questioned further she said she needed to reinforce her understanding of how to obtain eigenvalues. She felt that at school level calculator 'magic' (obtaining values without understanding what was going on) was acceptable, but that now she wanted to be quite sure she understood what the calculator was doing. She thought it was an advantage to have come across matrices at school, even at the simple level of notation, addition and multiplication, because "it links up and I can get on with the more difficult bits."

4. Student D was not a focus for observation because he did not appear to have difficulty with the work, however one of his contributions to class discussions is included here to provide background to student E's difficulty. Working on a problem which gave an absorbing Markov chain with the state diagram shown in Figure 1, the tutor obtained Table I and pointed out the effect of row and column order on elements - the blocks of probabilities and zeros, and the identity matrix.

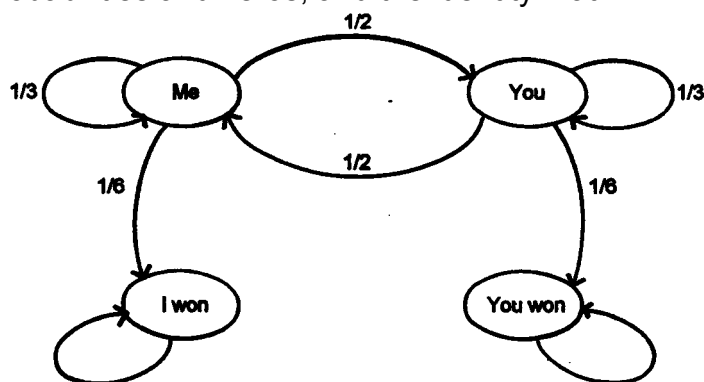


Figure 1

		non- absorbing		absorbing	
		M	Y	I _w	Y _w
non-absorbing	M	2/6	3/6	0	0
	Y	3/6	2/6	0	0
absorbing	I _w	1/6	0	1	0
	Y _w	0	1/6	0	1

Table I

D asked the tutor: "Does the bottom left do all the absorbing?" The tutor replied: "Yes, this is where it all happens. The eigenvectors are not interesting, it's the change of state which is." There was then some work with manual and calculator iteration followed by discussion. D then observed: "I saw the move from Me/You to I_w/Y_w from the diagram and tied it to the table. I would have chosen the same arrangement for the table, but for the wrong reasons! I think it looks better." The tutor went on to discuss the block structure of the table and the varying number of non-absorbing and absorbing states which could be

present. He finished by showing that all cases could be represented by the simplified matrix:

$$M = \begin{pmatrix} A & 0 \\ B & I \end{pmatrix}$$

where A , 0 , B and I represent the corresponding blocks whatever the number of constituent elements.

5. The validity of using the constituent blocks of an absorption table as elements for matrix multiplication was not questioned at the time it was introduced. However later, when they were working on their own, student E asked the tutor for further clarification of this technique. She listened as the functions of the different blocks were explained but made no input until the original example, with numbers as elements, was considered. She then tried to go back to the longer method instead of the shortened form. The tutor explained that he wanted to continue using blocks, without the necessity of considering all the elements of the whole matrix.

Analysis and Discussion of Observations

Reversion to familiar methods

In observations 1, 2, 8 and 5 students are seen to go back to familiar methods whilst encountering new processes. However, there seem to be differing reasons for returning to previous methods. Students A, B and E fail to understand or succeed with new processes and attempt to employ those they feel they can handle, whereas student C reverts to a method in which she was confident in order to confirm her new skill with a graphical calculator: "It's still quite fresh, this stuff; we still want to know what we're doing." Thus while three observations appear to support the original hypothesis, one draws attention to the student's need to link new work with previous mathematical understanding. These two fundamentally different reasons for reversion to old methods are considered separately.

Familiar methods as an 'escape' from new approaches

Rejection of new processes in favour of familiar ones which have worked for similar problems is understandable - after all, if successful this procedure is labour-saving. In Piagetian terms assimilation to an existing schema is easier to achieve than modifying a schema to accommodate new factors.

Whilst some students are motivated to learn new processes by the simple fact that they are new, others need a more pragmatic form of motivation - the new process must be essential for further progress. In fact this was true for all three problems, but not recognised by the students. Student A, in particular, fails to understand why his differentiation problem cannot be solved efficiently by familiar methods.

Investigations of novice and expert methods of tackling problems, such as Chase and Simon's study of chess players (Kahney, 1986, pp. 103-105), provide evidence of differences in approach and processes used. Experts classify problem configurations in terms of underlying principles, and can reformulate problems to make efficient use of known solutions. Novices improve their performance as they learn to analyse problems and recognise patterns of increasing complexity.

So when expert mathematicians pose problems they are aware of the 'deep structure'

which determines appropriate methods of solution. Students may more readily recognise 'surface structure' and attempt to 'match' this to familiar methods.

A careful choice and presentation of initial problems can draw attention to the 'deep' factors which make previous methods inappropriate. Consistent use of new methods then allows students to gain insight into the deep structure of more obscure problems.

Bruner described such organised support of student learning as *scaffolding* and developed the idea from Vygotsky's concept of *zones of proximal development* (ZPD). In the first example the problem is beyond the student's ZPD because he has no previous knowledge of parametric equations. In addition his return to the standard method of differentiation indicates that a 'scaffolded' approach is required before he comes to a deep understanding of why a new mathematical approach is necessary. Student A has focused on finding a solution to a particular problem in preference to learning a new mathematical technique.

Student B's 'escape' from mathematics is not immediately obvious to an observer and not recognised as such by the student. He sets out to revise geometry within the context of a geometry software package. The specific program ensures some geometrical content, but his activity centres on computer skills rather than geometrical understanding, and the problem which arises is voiced in computer terminology - what is wrong with the macro? - rather than geometrically - what is the geometry of the situation?

Van Glaserfield describes the teacher's role as:

to help and guide the student in the conceptual organisation of certain areas of experience. Two things are required for the teacher to do this: on the one hand, an adequate idea of where the student is and, on the other, an adequate idea of the destination.

(Van Glaserfield, 1987)

In this case, the student, who is in effect his own teacher, knows where he is with computers but has lost sight of geometry as his 'destination' and focuses instead on solving a specific problem.

Student E is not trying to use an inappropriate mathematical method, but fails to organise her understanding so that she can 'short-cut' the 'old' method by using blocks as elements. Her understanding of the process could be said to remain at novice level, whilst the expert restructures the problem by working with blocks.

The 'ability to curtail the process of mathematical reasoning and the system of corresponding operations (the ability to think in curtailed structures) is recognised by Krutetskii as one of the mathematical abilities of school-children (Krutetskii, 1976, pp. 263-275). However, he draws attention to research which shows that, although mathematically gifted pupils often 'short-cut' a process very rapidly, curtailment is normally achieved gradually, as a result of practice and growing familiarity with the process. Curtailment needs to be based on a deep level of understanding so that the complete process underpins the shortened version and can be recalled or constructed when necessary.

Mason sees the connection between the surface structure of concrete examples and the deep structure represented by symbolism in terms of a spiral which allows movement in both directions and says:

It seems to me important to be able to go back and regain confidence in why it works, by tracking back down the symbolising spiral.
(Mason, 1987)

Student E may need to continue to work with the complete process in order to gain a deeper understanding of the shortened version; on the other hand her difficulty may arise from the hierarchy of symbolic representation which has developed and which Goldin (1987) suggests requires higher levels of reasoning.

Familiar methods used to reinforce new approaches

Student C returns to 'hand-calculation' of eigenvalues in order to reinforce her familiarity with a complete process, so that the curtailment achieved by using a calculator represents a deep level of understanding and not simply a rote or 'black-box' method. Her comments about the use of calculators in school being magic show that she recognises this possibility. They also present a different facet to the instrumental/-relational learning debate, for an early instrumental understanding of the technique allowed her to concentrate at a later date on developing a relational understanding of the process it represents.

In addition, student C refers to 'linking in' previous knowledge, which is very different to 'reverting to' familiar processes. Another student also 'linked back' to his understanding of matrix multiplication when he came across the alternative convention for setting this out. His correct interpretation of the new format depended on referring back to a deep understanding of a process which he usually accepts as valid without further thought.

In effect, both these students are able to resolve potential difficulties and to accommodate new mathematical methods without the additional challenge of unfamiliar techniques.

Negotiating new mathematical territory

Student D's contributions to classwork record his recognition of new mathematical representations. His comments could also be interpreted as testing his own understanding of their significance and thus making links between symbolic representations and their mathematical 'roles'. For example he identifies and later checks the absorbing role of the lower left block of values in the table. He is actively seeking connections between mathematical concepts, and thus deepening his level of understanding.

Some Implications for Undergraduate Mathematics

New techniques used for problems that are similar to others already familiar to students seem likely to be rejected in favour of previous techniques if the need for a new approach is not immediately obvious. This reaction is predictable and may be counteracted by drawing attention to circumstances when the familiar processes fail. However it is probably more productive to use such circumstances as the starting point for the new techniques.

New mathematical methods involve a process of accommodation, which can take time, and which certainly involves the student in making connections with previous work. Discussion of such connections as a class activity might encourage the process of accommodation.

The curtailment of mathematical reasoning is a consistent feature of mathematics

which is not usually made explicit by tutors. Understanding the importance of curtailment of reasoning and recognising that practice may speed the process might help students who encounter difficulty in curtailing mathematical operations. They could be given an insight into this aspect of learning during a study skills course.

Instrumental understanding, which allows students to implement mathematical techniques, also leaves them free to concentrate on developing relational understanding of processes. Instrumental followed by relational understanding could be an effective teaching approach, particularly where computer or calculator techniques are involved.

The Value of Taking a Piagetian Viewpoint

Whilst the hypothesis was to some extent supported by my observations, the real value of taking a Piagetian viewpoint lay in the incomplete structure it provided for interpreting the difficulties observed. The processes of assimilation and accommodation provided a broad first level of classification, but then I was forced - by the lack of a more detailed structure - to consider other explanations and theories in the light of my own observations, but within the framework of Piaget's theory.

These drew my attention to two particular aspects of my observations:

- a possible correlation between new forms of notation or levels of symbolism, and the onset of difficulties (parametric notation, geometric notation conventions, and matrices as elements in a higher order matrix, are the examples here)
- the relationship between depth of understanding and curtailment of mathematical processes

However none of the theories to which I referred touched on a third aspect:

- the pursuit of solutions to problems rather than understanding of mathematical processes (demonstrated by students A and B and, to a lesser extent, by student E).

Conclusions

Change in notation appears to be a common feature in the difficulties observed in this study, and the degree of notation employed in mathematics makes symbolism a particularly interesting area. Howard Gardner describes *streams* of symbolic development occurring in four *waves* (Gardner, 1993, pp. 78 - 76), referring to *second-order* symbolic notation, such as the use of '5' or '+'. Kaput (1987) and Goldin (1987) both explore the understanding of more complex representational systems, whilst Commons *et al* (1982) base their levels of reasoning on an analysis of the systems of representation.

The apparent link between changes in notation and difficulties with the mathematics suggests that it might be equally productive to turn this approach around and ask whether changes in notation signal increasing complexity and the need for higher levels of thinking. The need for more complex notation in order to curtail mathematical processes both supports this view, and indicates that problems arising from the use of new

notation may be rooted in a superficial rather than deep understanding of the underlying process.

The students who were more intent on solving the actual problems than thinking mathematically lead me to query their perception of the relationship between problem-solving and mathematical thinking.

A further question is whether the problem-solving approach in mathematics promotes the accommodation of new mathematical ideas as well as deepening understanding of familiar ones by the assimilation of elaborated examples. This again raises the question of which cognitive features lead to a need for accommodation rather than assimilation, and reintroduces the idea of levels of thinking and degrees of abstraction.

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BOOK REVIEWS

Terezinha Nunes & Peter Bryant:
CHILDREN DOING MATHEMATICS
Blackwell, Oxford (1996), 268 pages

Children Doing Mathematics must surely be essential reading and reference for anyone with an interest in mathematics education for young children. The book is very readable, dealing with complex issues in a clearly-structured and engaging manner. It is well presented and has a delightful cover that will be the envy of many other authors in the field of education!

Nunes and Bryant do not just provide a useful review of significant research related to children's understanding of number, counting, numeration systems, measurement systems, number operations and rational numbers. They bring to this review their own insights and research and formulate a coherent and convincing theoretical framework for describing the complex ways in which children's mathematical understanding might develop.

This framework has three main parts to it. First, they take as a premise the view that children's understanding of mathematics is generative - in the sense that they can generate knowledge that they have not learned or been taught directly - and that their understanding of any mathematical idea changes many times and in many ways as they grow older. For each mathematical concept considered there is evidence provided here that children have some understanding of it at a much earlier age than that at which it is introduced in school, and also that their understanding is never a case of all-or-nothing

Second, they draw on the ideas of Gerard Vergnaud to discuss mathematical development in terms of the relationships between the logical invariants associated with mathematical systems of representation (e.g. .. various mathematical symbols) and the different situations in which these can be used - it is the connection between the invariants in the 'problem situation' and those in a mathematical procedure that determines whether it is an appropriate procedure for the problem. The word 'doing' in the title indicates the authors' conviction that children develop understanding of mathematics by solving problems that put them in a position where they have to make connections (via mathematical symbols) between the invariants in the situation and the logical invariants of mathematics.

Third, the theoretical framework embraces the view that mathematics is a socially defined activity. For example, the authors draw convincingly on the studies of Brazilian street vendors (Nunes and others) to demonstrate that children's responses to situations requiring mathematical reasoning are determined to some extent by the social context. In an informal, real-life situation a child might employ *ad hoc* reasoning and intuition to solve a problem, whereas in a formal classroom context they will abandon thinking in favour of trying to remember roles.

Although the main focus of the book is the psychological basis of children's understanding of mathematical ideas, some clear implications for teaching are drawn and these are all the more convincing because of the sound basis of theory and analysis from which they emerge. For example, in discussing multiplication and division the authors draw this general conclusion (p.201): "If we concentrate in mathematics instruction on teaching techniques, and pay only token attention to the relationship between the model and the situation it mathematics, we will create a divorce between the knowledge of techniques and the awareness of meaning." At a more detailed level, the review of young children's early understandings of rational numbers leads to the conclusion that fractional language and representation should be introduced through solving division (sharing) problems with continuous quantities, rather than through the conventional, static part-whole representations.

The word 'mathematics' in the title is a little misleading, since geometric concepts are hardly touched on at all, but the analysis of young children's understanding of all aspects of numeracy is certainly comprehensive.

This book is highly recommended as essential reading for all colleagues involved in mathematics education teaching or research. Although some aspects of it may be a bit too erudite for the majority of initial training primary students, they should nevertheless know of the book's existence and be encouraged to refer to it when undertaking studies of mathematics for young children. It will be of particular interest and value to those student-teachers specialising in mathematics (or developmental psychology). It will certainly be a valuable source of reference for many higher degree and research students.

*Reviewed by Derek Haylock
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Christine Hopkins, Susan Gifford & Sandy Pepperell (eds)

MATHEMATICS IN THE PRIMARY SCHOOL

A Sense of Progression

David Fulton, London (1996), 196 pages

This book should prove to be a popular resource for primary student-teachers. It is written by the team of tutors at the Centre for Mathematics Education at the Roehampton Institute, with two of the editors being practising primary deputy-headteachers. This blend of experience in primary-school-teaching and teacher-training has contributed to a book that speaks confidently and reliably about approaches to teaching mathematics in the primary school.

The book is based on the National Curriculum for England and Wales, with section headings corresponding to the sections of the programmes of study. This should make the book appealing to students and teachers in these countries. (The authors are presumably prepared for limited overseas sales and for a substantial rewrite when the National Curriculum is next revised!)

Two interesting features of the book are the sensible use of examples of children's work and some interesting 'case studies' that describe work undertaken by teachers with various groups of children. Most sections end with a 'glossary' that defines or explains some key words - although it is a little odd that the shape-and-space section does not include a glossary, given the extensive range of technical words used here. The problem with providing a glossary is that the authors leave themselves open to criticism from picky mathematicians about the accuracy of some of their definitions (such as that for 'function').

The authors have clearly set out to provide a comprehensive textbook that will meet all the needs of initial training student-teachers for planning and preparation of their mathematics teaching in schools. Inevitably within the limited scope of 196 pages some matters are therefore dealt with rather superficially. For example, student-teachers may find insufficient guidance on the practicalities of teaching methods of calculations. The final section dealing with planning, assessment and classroom organisation is fairly lightweight, given the substantial issues that might have been addressed here. For example, there is little consideration of the range of purposes of assessment in mathematics teaching and the issues related to differentiation in planning children's classroom activities. Another small but significant criticism is that the index could be more comprehensive and detailed.

On the whole this is a useful, practical and up-to-date book. It is nicely presented with a good balance between text and illustrations. It disappoints in that there is little in the way of original insights into children's learning or fresh perspectives on approaches to teaching mathematics. It is- actually quite difficult when you get to the end to identify any overall themes or to say what it is that makes the book distinctive. But it is accessible; it is relevant to student-teachers' immediate needs; and it promotes sound principles for effective practice in the classroom. I will certainly recommend it to my own primary students.

*Reviewed by Derek Haylock
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David C. Johnson & Alison Millett (eds)

**IMPLEMENTING THE MATHEMATICS NATIONAL CURRICULUM: POLICY, POLITICS
AND PRACTICE**

Paul Chapman Publishing, London (1996), 157 pages

Implementing the Mathematics National Curriculum is the first book of the series 'New BERA Dialogues' (general editor is Donald McIntyre). The planned series serves BERA's primary purposes of '... fostering educational research of high quality and demonstrating the value of good educational research to as wide an audience as possible ...' (p. ix). It achieves these purposes in terms of the quality of the research although I have some doubts as to whether 'as wide an audience as possible' is likely to include many practising teachers, which is unfortunate as there is much food for thought and action contained within this volume.

The authors are all very experienced and respected Mathematics educators; in particular, the book benefits enormously from the distinguished contributions of Professor Margaret Brown. Her personal involvement, in various capacities, throughout the development of the National Curriculum for Mathematics, provides valuable insights to the historical and political contexts and to future possibilities for 'defining a research and evaluation programme to begin in 1996, or no later than 1997' (p. 125). This should, in my opinion, be essential to usefully inform the next review of the National Curriculum.

The book comprises six chapters, each of which may be used independently. There is a useful glossary of acronyms which is essential for any reader unfamiliar with such terminology. Chapters 1 and 6, taken together, form an excellent, comprehensive review of developing policy and changing concerns in the curriculum and its assessment, during the introduction and implementation of the various forms of the National Curriculum for Mathematics, from its inception in 1989 to the current 1995 Order. It is disappointing to learn that many of the research recommendations described in chapters 2 to 5 were 'ignored or rejected by the Dearing review' (p. 121), such that the current 1995 Order is 'mainly a cosmetic slimming down of the previous version with minor corrections and new, but not overly helpful, PoS for each key stage', (p. 124).

In chapter 2, David Johnson and Alison Millett detail the implementation and selected results of a 2-year research programme designed to investigate the four areas of special concern in the implementation of the National Curriculum across Key Stages 1 to 3 that have been revealed by NCC and HMI, namely:

- Difficulties encountered in the implementation of selected topics
- Inappropriate progression in the POS and the ATs
- Implementation of AT 1 (Ma1)
- Effectiveness of the PoS for in-school planning

The research is reported in full in Askew *et al* (1993).¹ The research findings from the four studies resulted in a series of 17 recommendations '... put forward for discussion and debate' (p. 50). These recommendations were presented to the Dearing review although

several were either ignored or rejected.

Alison Millett and David Johnson address issues associated with the use of commercial Mathematics schemes in chapter 3, concluding that teachers '... may exacerbate (their problems) by turning uncritically to commercial schemes ... (they) ... may remain unfamiliar with the NC, and may never, as a result, achieve 'ownership' of it and be able to internalise it and adapt it to their own use' (p. 78). The authors note that mathematics co-ordinators and heads of department need time to facilitate development of appropriate schemes of work that do not rely exclusively on commercial schemes.

Chapter 4 focuses on teachers' professional development. Stephanie Prestage confirms the conclusion already stated in the previous chapter that '... what became evident in the project and crucial to the teachers was the need for time to make sense of an external curriculum and time to integrate it into a particular school's needs and resources' (p. 98). Conclusions which advocate time being 'found' for teachers to work in these areas will undoubtedly be welcomed by the teachers, but need to be received equally positively by those in a position of being able to provide them with that time.

Mike Askew addresses issues surrounding Ma1 (Using and Applying Mathematics) which became evident from interview data, in chapter 5, concluding that it seems likely that Ma1 is being only partially implemented, due to teachers' particular interpretations of this attainment target. He recommends that teachers are encouraged to '... exploit the potential for such beliefs to be challenged, extended or developed' (p. 112).

The research results demonstrate a clear need for both further research to be undertaken prior to any more changes in the National Curriculum and for appropriate professional development to be made available to teachers in post throughout all three key stages. The conclusions and recommendations discussed in this volume provide a solid basis from which those in positions of power and influence could develop a proactive stance towards future positive developments in Mathematics education. Indeed, many of the recommendations are appropriate for other areas of the curriculum. Recommendations to assist practising teachers' professional development are pertinent: dissemination of such recommendations to appropriate audiences may prove more problematic.

There is a useful index and a valuable bibliography which could be helpful to tutors in initial teacher education, trainee teachers, researchers, practising teachers and policy-makers alike. Chapters 1 and 6 are particularly useful for all interested in the myriad of activities surrounding the introduction and implementation of the National Curriculum for Mathematics during the past seven years. Although the research was completed three years ago, the results and recommendations seem as valid now as they were then.

This is a well-written, clearly explained, jargon-free book, which I recommend to researchers, teachers, student-teachers, curriculum developers, professional developers and even, or perhaps particularly, politicians! It deserves to play a prominent part in the continuing debate on improving Mathematics teaching and learning.

Reviewed by Anne Sinkinson University of East Anglia Norwich

¹ Askew, M. et al. (1993). *Evaluation of the Implementation of National Curriculum Mathematics at Key Stages 1,2 and 3. Volume 1: Report. Volume 2: Appendix.* London:SCAA.

Derek Haylock and Anne Cockburn:
Understanding Mathematics in the Lower Primary Years
(revised and expanded edition)
Paul Chapman Publishing (1997) 208 pages

Recent comments from HMI and the publication of proposals for an Initial Teacher Training Curriculum for Primary Mathematics have focused attention sharply on the need for teachers to possess a clear understanding of basic mathematics if they are to help children effectively to develop related skills and concepts. *Understanding Mathematics in the Lower Primary Years*, which is a revised and expanded version of the authors' earlier book *Understanding Early Years Mathematics*, will prove to be a valuable source of support in this respect for both existing practitioners and student teachers. There are also innumerable examples of straightforward explanations and anecdotes which will be a welcome resource for tutors in initial teacher training.

In addition to considerably revising existing materials, the authors have added three new chapters, covering aspects of number patterns, data handling and problem solving. Each chapter concludes with suggestions for relevant classroom activities, usually differentiated, a useful summary of the key ideas covered and suggestions for further reading. The importance of language and communication is constantly reinforced and is supported by examples of discussions between children and teachers, students and tutors.

The authors begin with a warning that this is not a book which the reader will be able to 'canter through'. This may be quite true, but this particular reader thoroughly enjoyed a brisk and lively walk, taking pauses now and then for reflection and digestion. Much of the work is inspired by a period of time which the writers spent with a group of teachers who discussed their personal experiences and mathematical understandings. As a result the material is firmly based within the context of lower primary classroom practice.

Some quite complicated mathematical principles are explained in a style which is essentially accessible, achieving a fine balance which informs and extends but never patronises or intimidates. Sound explanations clarify the reasons for the importance given to individual strategies. The authors explain *why*, for example, it is important to use a variety of terms to describe the various contexts of subtraction. If children have never developed an understanding of 'the difference between' two numbers within the comparison structure, how will they cope with later encounters with calculations such as '6 - (-3)'?

The new chapter on Number Patterns and Calculations is particularly successful, with detailed coverage of how an increased use of the 'hundred square' can assist in the development of number dexterity as children explore and use the patterns discovered to deal with problems of addition and subtraction. To work with bigger numbers, bridging the hundred landmark, the square is expanded to a two-hundred-grid - an example of the type of simple and effective idea included by the writers which seems so obvious, yet is rarely found in published schemes.

Another useful feature of this chapter is the section describing some of the errors commonly made by children wrestling with vertical algorithms before developing a secure understanding of place value. Student teachers will find particularly helpful these examples of the insight into children's thinking afforded by an analysis of incorrect

responses and the contribution that this can make to diagnostic assessment.

The National Curriculum for mathematics begins with coverage of the skills of using and applying mathematics, but in many ways it is appropriate that the authors have considered these aspects at the end of the book. In this final chapter, several themes which have recurred throughout the text, including the importance of language and visual information, the concepts of transformation and equivalence and the ability to organise and internalise procedures are gathered together in a complete process of 'Thinking Mathematically' as problems are solved in both real and abstract contexts. Practical examples provide illustration, but here, as in other areas of the book more suggestions specifically designed to appeal to the younger 'early years' children would have been welcome.

There have been few mathematical texts which cause readers to smile, but one of the great strengths of this book is the consistent good humour. Hiding in the explanations of more abstract concepts are unexpected images, such as the 'two thousand sets of zero elephants placed inside these brackets []', which surprise and entertain, and so render difficult ideas memorable.

On reaching the last page of this text, my immediate reaction was to find a sticky name label and place it firmly in the middle of the front cover. This is a book of which I do not intend to lose track!

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