CONTENTS

MER11 Editorial

Welcome to this special edition of *Mathematics Education Review*. Authors were invited to submit articles about their practice in initial teacher education (ITE) in order to begin to disseminate this practice. We have used these case studies to begin to analyse and develop a sense of the pedagogy of teacher-educators.

Prologue: Towards a pedagogy of teacher education: a model and a methodology

Teaching requires a myriad of knowledge and skills, knowledge about students, systems and structures, knowledge about styles of teaching and learning, knowledge about management, resources and assessment as well as knowledge about the subject. Existing research in the area of teachers' knowledge, offers definitions of professional knowledge as well as explanations for the different forms of knowledge that a teacher holds (Shulman 1986; Wilson et al. 1987; Brown and McIntyre 1993; Aubrey 1997; Banks et al 1999). Elsewhere we have defined a model for considering the development of subject matter knowledge necessary for teaching mathematics where personal subject matter knowledge and professional content knowledge of teachers are mediated by deliberate reflection in order to create a more fluid and connected personal understanding of mathematics needed for the classroom (Prestage and Perks, 1999). This model can be paralleled for the work of teacher-educators, and influences the way in which we analyse the papers in this journal in the closing paper.

The initial construction of the model starts from an awareness of the beginning stages with our PGCE students. ITE students arrive with a certain amount of personal subject knowledge (*learner-knowledge*) that enables them to answer mathematical questions. Their subject knowledge is ill-connected and they have to work on this when planning for teaching (Perks and Prestage, 1994). They also bring with them their personal beliefs and certain characteristics of 'being a teacher'. Through the PGCE year they gain different knowledge and understandings of other *professional traditions* - some global like the National Curriculum, the Numeracy Strategy and the examination system with all their attendant exemplar materials, and some local traditions gained from particular school settings such as schemes and textbooks - the ways in which national policies are translated in different settings. Learner-knowledge and professional traditions merge in the first instance to create classroom events for

others to engage with learning mathematics, figure 1.   These classroom events, and the lesson plans which precede them, offer the first evidence for pedagogical content knowledge (Shulman, 1986).
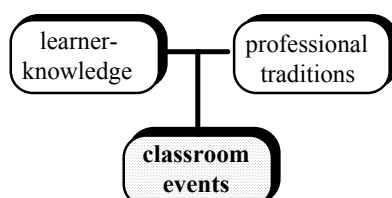

figure 1

Reflection upon these events (reflection-upon-remembered-action) leads to the beginnings of some practical wisdom as the students discover that telling doesn't work, all learners are different, certain misconceptions affect early learning, efficient algorithms are not easy to remember, etc. Students' lesson evaluations can reveal this practical wisdom (e.g. Perks, 1997) and is most in evidence when it is used to reconstruct lessons, explanations and demonstrations and enables the students to adapt activities from the professional traditions to suit their particular circumstances, figure 2.
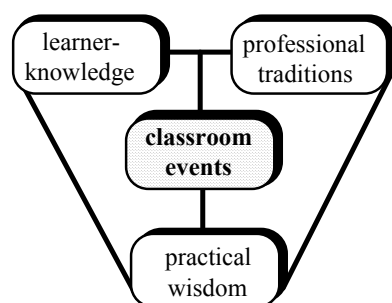

figure 2

Our beliefs about teaching concur with those of Buchmann (1984), that teachers need a rich and deep understanding of their subject in order to respond to all aspects of pupils' needs.

> *Content knowledge of this kind encourages the mobility of teacher conceptions and yields knowledge in the form of multiple and fluid conceptions.*   *(ibid. p.46)*

We believe therefore that 'good' teachers reflect upon the classroom events at a further stage, i.e. to reconsider their own personal understandings of mathematics, to reflect upon the 'why' not only of teaching but also of mathematics. They come to own a better personal knowledge of mathematics (*teacher-knowledge*) that allows them not only to answer the questions correctly but that also helps to build a variety of connections and routes through that knowledge, and that provides answers to 'why' something is so (Prestage, 1999). It is our contention that only when such subject knowledge is

informing classroom practice that the real needs. of learners and the challenge of mathematics are addressed.
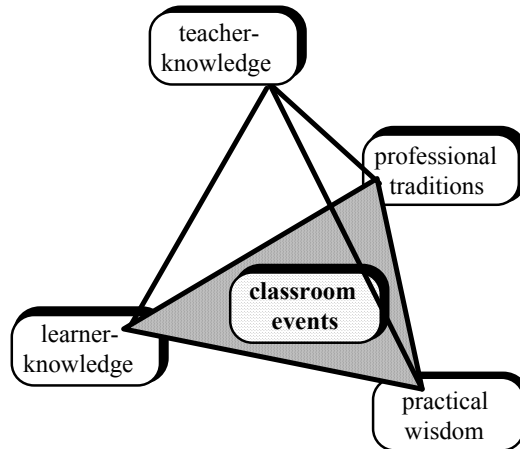

figure 3

What then are the implications of this for thinking about a pedagogy for teacher education? Is there an equivalent *teacher-knowledge* which informs our practice? We will construct a parallel model to the one above but take the learner-knowledge for a teacher-educator, figure 3, as subject knowledge plus all other aspects of general professional craft knowledge.
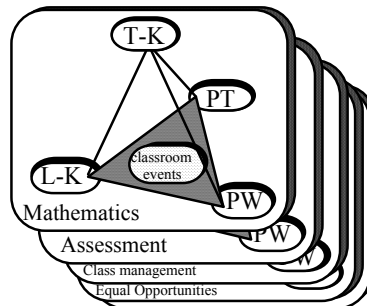
figure 4: Learner-knowledge for the teacher-educator

I arrived to my current job as a teacher-educator with all aspects of the above in place. This was my learner-knowledge for being a teacher-educator. I had both learner-knowledge and some teacher-knowledge for mathematics subject matter, I understood the professional traditions and held a certain amount of practical wisdom. I also held a variety of other professional knowledge about general teaching matters. These then formed the basis for me to reflect upon and analyse and synthesise for others to come to know about teaching mathematics.

What then are the professional traditions of the teacher-educator? What is the practical wisdom? Can we identify these and make explicit the teacher-knowledge?

The professional traditions of our current profession emerge from personal experiences, education and training, the current government and TTA policies, the mathematics and ICT ITT National Curricula (DfEE, 1998) as well as Ofsted (1999) criteria against which judgements are made. Practical wisdom can be defined as considering what the students need to know and how sessions might be constructed for them so that they engage in the ideas.

figure 4

figure 5

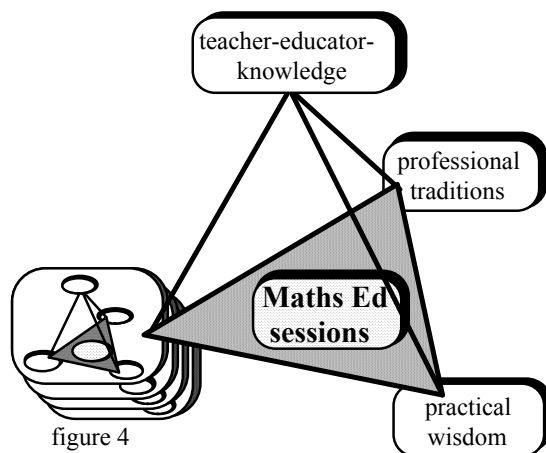Currently there are few places to read about practice in ITE in order to improve our practice by considering what others do. We sneak a few articles into mathematics teaching journals, as do others, and from these we come to know our job better. We believe that just as teaching mathematics needs fluid and connected knowledge of mathematics (teacher-knowledge) so too as mathematics educators we need an articulated, fluid and connected understanding of teaching mathematics education – the teacher-knowledge of mathematics education.

Hence this special edition of MER. Here are opportunities to share the professional wisdom of teacher-educators and to use these to reflect upon our own practice and to reconsider our own understanding of our knowledge base.

Shulman (1986; 1992) has articulated a variety of ways in which case studies can contribute to our knowledge about teaching, the case study allows for a detailed picture of the particular which may allow for more general principles to emerge. Such case studies enable research to be based within an interpretative research paradigm, an enquiry carried out to understand what is happening, "striving to investigate without disturbing" (Bassey 1995, p.6).

This journal offers a sample of case studies from teacher-educators' practice in initial teacher education. This allows us to study in Bassey's terms, a "singularity". With a progressive focusing on the data (ibid. p.7) we hope a theory will emerge which will be grounded in the case study data investigating

the nature of pedagogical content knowledge in teacher education. Practically, each of the case studies will provide an illustration of a specific instance. By reflecting upon these with reference to a synthesis of learner knowledge and practical wisdom of other mathematics teacher-educators, the equivalent of a teacher-educators teacher-knowledge might emerge. The aim of this process is to make explicit the multiple and fluid conceptions of the art of teaching mathematics education.

Once we have a shared understanding from the 'wisdom of practice' (Shulman, 1986) we may be better prepared to move towards an articulation of a pedagogy for mathematics teacher education.

References

Aubrey, C. (1997) *Mathematics Teaching in the Early Years: An investigation into teachers' subject knowledge*, London: Falmer Press.

Banks, F., Leach, J. and Moon, B. (1999) New Understandings of teachers' Pedagogic Knowledge in Leach, J. and Moon, B. (eds.) *Learners and Pedagogy*, London: Paul Chapman Publishing.

Bassey, M. (1995) *Creating Education Through Research*. England: Kirklington Moor Press (in conjunction with BERA).

Brown, S. and McIntyre, D. (1993) *Making Sense of Teaching* Buckingham: Open University Press.

Buchmann, M. (1984) The priority of knowledge and understanding in teaching in Katz, L, and Roth, J. (ed.) *Advances in Teacher Education* (Vol 1) NJ: Ablex, pp.29-50.

Ofsted, (1999) *Secondary ITT Subject Inpections: Guidance for Inspectors: 1999/2000*, The Office of Her Majesty's Chief Inspector of Schools in England.

Perks, P. and Prestage, S. (1994) Planning for learning, Jaworski, B. and Watson, A. (eds.) *Mentoring in Mathematics Teaching*, London: Falmer Press.

Perks, P.A. (1997*) Lesson Planning In Mathematics: a study of PGCE Students*, unpublished PhD Thesis, University of Birmingham.

Prestage, S. (1999) *An Exploration into Teachers' Subject Knowledge*, unpublished PhD thesis, University of London.

Prestage, S. and Perks, P. (1999) Questions about subject knowledge: learner-knowledge and teacher-knowledge in Bills, E. (ed.) *Proceedings from the 4th British Congress of Mathematics Education July 1999*, University College of Northampton.

Shulman, L.S. (1986) Those who understand: Knowledge growth in teaching, *Educational Researcher*, 15, 2, pp.4-14.

Wilson, S.M. Shulman, L.S. and Richert, A.E. (1987) '150 different ways' of knowing: Representations of knowledge in teaching in Calderhead, J. (ed.) *Exploring Teachers' Thinking*, London: Cassell Education.

Using Cabri-Geometre to support undergraduate students' understanding of geometric concepts and types of reasoning

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This article describes the development of two sessions with first year B ed. mathematics students, using dynamic geometry software to develop geometric reasoning. Although Cabri is not ideally suited to the primary school I have chosen it as an example of a computer representation that adds significantly both to understanding of mathematics and to how mathematics is learnt. Before describing the sessions themselves I will analyse their design using the social constructivist framework developed by Simon (1994) which extends theories of learning mathematics into learning to teach mathematics.

Teaching Mathematics and Teaching Mathematics Education

Maths BEd. sessions necessarily involve learning for two different purposes - mathematical and professional. These students are experiencing a major change in their understanding of what doing mathematics is, just moving from school to university mathematics, from practical/empirical to abstract/deductive, and from maths as passive and tidy to problematic and personally demanding. Simon (1994) describes how experiences of *learning mathematics* accompanied by reflection help teachers develop *knowledge about mathematics* and *personally meaningful theories of mathematics learning*, all providing the foundation for developing knowledge about teaching mathematics to school children. He stresses the recursive nature of these first three learning cycles:

> *what one understands mathematics and mathematics activity to be and what one identifies as important in the domain affects the nature of what is to be learned. Similarly, one's understanding of how one learns… affects one's subsequent learning of mathematics (p.81).*

On the one hand this gives a justification for the dual purpose of the course. The students' immediate and personal experience of finding that mathematical activity is not exactly what they thought stimulates interest in identifying new ideas about the nature of maths and of maths learning, while these in turn

develop their mathematics. University mathematics is not experienced as a yet more formal and distinct pinnacle of mathematics than secondary school maths, entirely separate from teaching primary maths, but as helping to define what the essentials of mathematics are, relevant across all ages.

The disadvantage of attempting both together is the lack of distance. Critical reflection is an essential part of the development from mathematical activity per se to knowledge about mathematics, and this requires both depersonalisation and abstracting from experience. Perks and Prestage (1999) describe using the square root algorithm as a prompt for discussion about learning mathematics - at the crucial point succeeding with the algorithm itself can be dismissed as irrelevant. This is acceptable as part of the didactic contract between PGCE tutor and student, but not between maths tutor and student. Identifying mathematical processes and ways of thinking is appropriate pedagogical knowledge for intending teachers but may come too early in some students' mathematical experience, leading them to use what should be descriptions as instructions, damaging both mathematically and for their understanding of what mathematics is (Schoenfeld, 1992).

Simon (1994) also notes that learning mathematics in a context where the ultimate goal is actually learning to teach may not produce much cognitive disequilibrium if students ascribe conflicts to the different contexts rather than changing beliefs about learning. Simon's description of learning cycles incorporates three phases - exploration, concept identification and application. He finds the interconnection of the learning *of* mathematics, *about* mathematics and *about* mathematics learning to be within the exploration and application phases, with the concept identification phase distinctive to each cycle. Simon describes the role of the teacher-educator as identifying and analysing the key concepts that he wants intending teachers to develop, and using these to generate appropriate activities. He stresses the importance of both pedagogical and mathematical analysis in planning mathematical activities, although the mathematics may serve as an exploratory phase for learning about mathematics education and the underlying pedagogical concepts not be made explicit to the students themselves. This contrasts with the Ofsted principle of stating your objectives to pupils at the beginning of each lesson.

Identifying Pedagogical Knowledge Relating to Cabri

When encouraging students to reflect on the nature of their learning I have found that learning geometry with Cabri provides a rich and motivating stimulus. Noss and Hoyles (1996) describe the computer as a *window* on pupils' learning. I aim to use Cabri for the students to observe their own learning. The broad key concepts I have chosen relate to the experience of

learning with Cabri, but also to how learning with Cabri is indicative of learning any mathematics. This pedagogical content knowledge underlies the design and presentation of all the activities, but only some is made explicit through discussion and reflection.

Distinctive features of Cabri

Cabri is a microworld, an open learning environment which is a medium for expressing certain geometrical ideas. Drawing packages help us to position perfect geometric shapes, but Cabri goes further in allowing us to interact with the shapes via their mathematical features and build up figures using geometric dependencies. A Cabri object can be picked up and dragged around the screen to any position compatible with its original definition, and all dependent objects will change position accordingly. This creates a new way of thinking about a geometric situation - instead of an ideal mental image, or a drawing coupled with an understanding of what is to be taken as typical, Cabri offers commands for creating a figure and the way it can be moved - a representation that is both immediate and responsive (Laborde, 1996). The dual aspects of interaction are vital: the mouse acts directly upon the visual image, but using the geometric commands gives the distance necessary for appreciating the mathematics (Hoyles, 1996).

Designing software involves making decisions about what mathematics will be represented and how. I want my students to understand that any use of software is associated with a particular view of mathematics and learning mathematics. Effective use in teaching depends on selecting and managing software appropriately to their intentions of what and how pupils will learn.

Different representations of geometry

Cabri provides students with a novel way to interact with some mathematics with which they already feel familiar. The students have seen primary lessons that introduce geometrical ideas as ways to describe and classify everyday shapes. They readily empathise with the principle of children building on what they know. Students are less familiar with the idea of children (and themselves) making understandable but erroneous inductions from their real world experience, particularly in geometry where visual features such as orientation or overall appearance are given a different emphasis than in mathematical definitions (Clements and Battista, 1992). Cabri helps to differentiate between visual features by giving direct control only over those which are also mathematical. As well as a means of producing many discrete examples, dragging is a dynamic form of rotation and controlled deformation, both valid strategies for deciding whether one shape is the same type as a familiar

example (Hasegawa, 1997). Such continuous variation has been found to be more helpful than exposure to a diverse selection of typical cases (Gagatsis and Patronis, 1990).

Ways of working

Any microworld is intended to be a problem raising and solving environment in which children construct their own knowledge. Tasks which leave no element of personal choice are simply animations, failing to use the potential of the software. By giving the students open problems to work on I intend them to experience both the satisfaction of personal control of their activity and the accompanying frustration, and to be able to consider when they may need freedom, help or motivation. I would distinguish two kinds of Cabri problem characterised by the nature of their goals and how students feel when they attempt it. The first is constructing a figure to a specification e.g. Can you make a square that stays square when I drag it? Can you make a figure that behaves just like this one? The end point is given in terms of the student's previous knowledge or verifiable computer behaviour; the problem is how to combine the computer tools to get there (and by doing so learn some mathematics). Students quickly engage with the task but may lose interest as strategies are unsuccessful. A second type is answering a question about geometrical relationships, e.g. what is special about cyclic polygons? This involves simultaneously deciding what is a valid or interesting endpoint in geometric terms and relating this to what is observable as Cabri behaviour. Students are most likely to feel uncertain during the initial engagement, not because they find the two representations hard to work with but because they are uncertain about what is of value in formal geometry.

The difficulties in this second type of problem illustrate the distinctions between problem solving and mathematics. Using the computer to work on a mathematical problem is not effective learning unless some external mathematical knowledge has been deployed in the solution. Similarly conjectures and explanations based on inductive strategies are not usually accepted by mathematicians until they are accompanied by deductive reasoning. Cabri is distinctive in that its commands are very closely matched to geometric strategies (although this still cannot prevent students trying them out at random to see what works), and that these include both inductive and deductive strategies. Hoyles sees Cabri's potential as providing "a set of three two-way mirrors - to and from induction/deduction, visual/analytic, drawing/figure." (Hoyles, 1996, p.98)

Cabri has a fairly restricted menu of commands which can be combined only in certain sequences that parallel Euclidean geometric constructions. This is

deliberate - to favour these strategies over others - and in this sense Cabri is a deductive environment (Bellemain and Capponi, 1992). It is also easy with Cabri to see what changes and what stays the same in a general configuration, especially if told what to look for, and this encourages inductive use more familiar to the UK curriculum. Familiarity with Cabri commands supports the children in using appropriate language to phrase conjectures but the structure of the figure is so far incidental to this approach. Either by then having to construct the figure according to geometric rules, or by seeing *how* the figure accommodates both change and invariance, pupils could further be assisted in reasoning deductively. Bellemain and Capponi compared children's Cabri strategies with their pencil and paper strategies and found little direct transfer between the two but observed that after using Cabri children who had previously offered no formal reasoning were able to attempt a deductive construction.

Dragging plays a key role in Cabri problem solving as an observable action with a clear formal interpretation. It is a contextualised way both of phrasing questions about generality and of verifying it. For Hoyles and Noss (1994) it was crucial that pupils understand that their constructions should not "mess up" when dragged. This gives pupils control of a geometric authority separate from the teacher. It is important however not to see the final criterion as characterising the whole problem solving strategy. Children and students often have several attempts at a construction, working backwards from a figure that looks right, and looking at the effect of placing objects by eye (Hazzan and Goldenberg, 1997). "Not messing up" is a Cabri goal that is not always necessary to the mathematical goal.

Computers and abstraction

In identifying Cabri's potential to offer geometric objects freed of physical distractions and give access to deductive reasoning I risk suggesting a desirable replacement of concrete with abstract representations. Wilensky (1991) questions the absolute nature of the concrete - abstract division, and argues that concreteness is "a property of a person's relationship to an object" (p.198). Objects can become concrete if we have many ways to engage with them and models to represent them. Cabri fulfils both these roles for geometric ideas, and does it well because it allows the learner to act on mathematically abstract ideas in a semi-formal way, making them accessible for exploration or problem solving activities.

Noss et al. (1997) discuss how mathematical meanings are constructed within such an *autoexpressive* environment - in which the only way to manipulate objects is to express explicitly the relationships between them. When students

reflect on and articulate relationships within the environment they are making a situated abstraction - a generalisation which is particular to that environment. The nature of what they are doing is mathematical, even if the final product is not phrased in mathematical terms. An environment which includes two representations - for Cabri, the appearance of the figure and the story of its geometrical construction - permits abstraction as making connections between the two rather than replacing one by another. This may in part explain the appeal of dynamic software to successful mathematicians: teachers' and some undergraduates' first reaction is one of real pleasure in at last finding a way to express their geometrical knowledge in a powerful representation. The connections are important in both directions.

The idea of a situated abstraction is useful to describe what is happening when you work with Cabri. There are distinct features of Cabri geometry which are different from mathematical geometry: it is easy to see a Cabri figure as an articulated mechanism that physically moves in time, rather than contingent examples (similar to interpreting a height-age scatter graph as someone actually growing). It is also tempting to consider that geometric objects and relationships depend on each other sequentially, whereas a mathematical relationship can be symmetric and it is often useful to exploit such a change of view. Using Cabri simply as a model of geometry is misleading, but the recommendation to explicitly compare the models begs the question of what true model of geometry children do have. I cannot justify the use of Cabri simply by its approximation to geometry. In contrast it is clear that Cabri does allow children to make situated abstractions concerning the connections between visual and language-based Cabri geometry, which are contextualised versions of the mathematical connections. If a model of true geometry does exist, it lies somewhere in this connecting web.

Principles of the sessions

After identifying these key concepts I decided to use tasks which

- showed how Cabri models geometric objects and reasoning about these objects
- permitted students to choose their own methods
- used precise geometrical language
- required students to work on geometry at the same level of formality as they are working on number/algebra in their undergraduate courses
- used geometrical relationships from Key Stages 1-3

The openness of the task could conflict with the principle of using precise language. I decided that the latter is important for a number of reasons. Undergraduates are continuously learning the implications of the mathematical and logical language used on the scope of a statement in their other courses, and should see that geometry can be treated with the same exactness. Actually using the mathematical meaning of familiar shape and space terms helps students to see that often we do not use it - approximations are sufficient for most purposes - and they may then appreciate why children need help in moving away from those approximations. An overemphasis on precision could inhibit students from working in their own way, and destroy the point of working in a Cabri context: if they are to make Cabri connections between the visual and the tool definitions, they need a mixture of languages as seems appropriate. My decision was to present tasks geometrically even where I feel that a more descriptive presentation would be easier (or more appropriate for primary children).

Another issue concerning use of computers is the extent to which I want to see transfer between computer use and pencil and paper geometry - do I actually require evidence of this through simultaneous use or follow up work? The relationship between Cabri and paper geometry is a feature of what I want the students to learn, and certainly more than an assessment device. Past experience shows me that students do not find it natural or easy to move from computer to paper. I do want them to have a way of presenting their finished work, both for motivation and to ensure that they have thought through and justified a solution. I decided to show them how to write text on the screen for this purpose, and the choice of when and if to make a paper diagram will be theirs.

Session One - The look of things

The first impression I wanted the students to get from Cabri was the idea of a general construction that represented all the different possible cases of certain rules. In this I want to avoid too much emphasis on building and moving, and to prepare the ground for the Euclidean menus. My introduction is away from the computer, handing out compasses and blank rulers and asking the students to draw with them. This at once gives me some feedback on who is thinking geometrically (very few of them) and provides me with raw material to use in a discussion of the difference between mathematical diagrams and pictures. Before starting this discussion, I choose one of the students' pictures which involves linked objects, such as polygons with shared sides (buildings) or circles and polygons (tree, face), and ask one student to describe it for the others to draw, without saying what it depicts. This again is to reinforce that we treat differently a mathematical and a pictorial description even when they are

of the same diagram. Points to raise in the subsequent discussion are the predominance of pictures, the scarcity of irregular or concave polygons, or triangles on their points, the convention that for example a window drawn approximately square is described as square. Putting together the many diagrams belonging to the same mathematical description, the group can eliminate any which are wrong and I can make the point that there are many valid drawings associated with one description.

After a quick whole group demonstration of the Cabri actions needed to construct a triangle, measure its angles and drag it into different appearances, I want the students to meet the idea of constructions that do and don't 'mess up'. I demonstrate this by constructing a circle and inscribed triangle. As I drag the circle the triangle moves too and I comment that it looks as if every circle has a triangle within it, indeed dragging the points around the circle gives many inscribed triangles, and ask - Does every *triangle* have a circle that fits around it? How can I show this on the screen? I hope to raise the question of whether the existing inscribed triangle is truly representative of *every* triangle. Starting with a fresh triangle I simply keep dragging a circumcircle into place, showing that now the circle is not linked to the triangle and cannot be linked unless I can work out where to put its centre and radius.

I then set the students, as pairs or individuals, the task of constructing a cyclic quadrilateral, and answering the question "Only some quadrilaterals that fit on a circle - what is special about them?", chosen as being open to different approaches - construction, empirical, conjecturing - and simple enough to allow me to troubleshoot. I then ask three or four students to repeat their conjectures to the group to illustrate different outcomes and phrasing. It is interesting that students who start with a circle and move vertices of a quadrilateral around the circle tend to comment on movement and individual angles - if you move this point its angle stays the same but these two others change. Those who adjust a quadrilateral to make it fit on the circle, comment more on the properties of the whole shape e.g. all the corners have to be the same distance from the centre. It may be that it is more striking to notice two results becoming true together than it is to observe that they are always true within a constraint.

The next exercise is intended to show another way of working with Cabri - from a prepared file. Having introduced the idea of general cases within a particular rule, I give the students a screen initially showing eight coloured squares and ask them to find out what shapes are there and to add text labels. Natural fiddliness will lead someone to drag the squares and quickly they discover that all but one deform, making a rhombus, rectangle, parallelogram, kite, trapezium, general quadrilateral and hexagon. The superficial objective here is unpacking the rule, with some revision of mathematical language, but

underlying this is the desire to challenge students' assumption that classifying shapes is unproblematic.   Asked to scroll down the screen, the students meet prepared text, starting with 'A square is a special kind of ------------.' and after suggesting different ways to fill the blank, they go on to complete 'Every --------- is also a ------------.', ' Some ------------s are ----------s but some are not', and 'If a shape is both a kite and a ---------, then it must be a ----------.'   Most students will find this difficult because they are not used to the language nor to considering shape classifications as subsets of each other. (For those students who do find it straightforward, drawing a Venn diagram of the quadrilateral relationships is challenging because of the empty intersections and multiple nesting.) Depending on the students' response to the task I finish in different ways. If they have been able to generate statements then I take some examples and ask how they can be verified using the Cabri diagram - what do you drag, and what should you see? If not, I ask them to think what was difficult about the task, and relate this to children's difficulties in having to use mathematical definitions that do not fit with their usual ways of understanding shape.

The final activity of this first session is again looking at a prepared file, in which I have constructed coloured objects but hidden all the construction lines. Text invites the students to find out which points can move, and to move point A and describe what happens to points B, C etc. I chose to place some points on a hidden circle and triangle, easy to spot and directing emphasis to the shape itself, not its interior (as children may do (Hoyles, 1995)). These are classic loci, but I really want the students to describe relationships between objects. For this I chose to include a line which is always parallel to AB and one which stays perpendicular, and points which are reflections in those lines. For surprise value I added some intersection points which exist only sometimes. One purpose of this activity is to give an example of a task which can be answered at any level of geometric knowledge, so after a quick exploration the students are asked to design their own hidden figures worksheet for children and record some different descriptions they might expect to get from a class. So far the activities have looked at the different appearances associated with geometric objects but this last activity is one where the geometric concepts are not actually visible but in the mind of the user. Having started by reminding the students of children's common reasoning about physical objects and diagrams and that this is not the same as their own knowledge of geometry, I want to be clear that Cabri figures are not true geometry either, but a way of helping to think about it.

Session Two - Problem Solving

In the next session I set two problems, to construct a square that would not mess up, and to find proofs of some geometric statements.

From past experience I know that students find it challenging to construct a square and that although they will define a square as having equal sides and equal angles, many will construct one from the endpoints of two perpendicular diameters of a circle, using a property of diagonals of a square that they did not consciously know. During the construction one possible role for the teacher is to circulate, reminding students that the square should not mess up when dragged, but I have found it more succesful not to re-emphasise this validation during the problem solving time. Instead I adapt Schoenfeld's (1992) interventions and ask students "What properties of a square have you been thinking about?", "What have you achieved so far?", "What are you trying to do now?", "Why?". At the end of the task I want to try to get students to reflect on their problem solving strategies, and maybe also whether these are influenced by the Cabri tools available. I have tried asking students to write about the mathematical stages of their solution but found their responses superficial, and not improved by providing prompts and exemplar phrases. Instead I show them how to replay their construction history and ask them to choose a moment to stop the history and write on the screen about their strategy at that time, repeating the questions above. This is a more well defined task and allows the students to choose something they consider significant. Rogers (1995) uses a similar technique, asking students to explain the logic of just one step in a given solution.

For the proof problem I gave a choice of 5 tasks to work on and suggested that they use Cabri or paper as preferred. This might be seen as preventing the establishment of common ground for the group to find generalities about proof strategies, but this seemed unlikely to produce worthwhile conclusions since they were not confident in proving or describing proofs. I am more hopeful that an individual will find similar techniques useful in tackling more than one problem, and that the association of a geometry technique with a Cabri action will both suggest its appropriate use and help distinguish it as applicable in different contexts.

The validity and rigour of a proof are socially determined qualities and students are rightly suspicious when tasks are presented as completely open so I start by giving illustrations of what might be considered a proof. The first is again to show the group a triangle, asking whether dragging the triangle proves that the angle sum is 180 degrees. It is interesting that even for undergraduates the idea that "it might just look right" is partly associated with measurement and display errors, not the principle itself. I can then deform the triangle by dragging a vertex onto the opposite side, watching the central angle increase as the two others decrease until the triangle disappears into a line segment - how could the angle sum be anything other than 180°? This movement both convinces and explains, meeting the two purposes of proof (Hanna, 1995). It is an example of

powerful reasoning from a generic example, and relies purely on observation not measurement but most students will not agree that it is a proof because it contains no formal reasoning.

The next example is to ask the students to prove that the exterior angle of a triangle is equal to the sum of the other two interior angles. The main point is that this feels like a proof because we can immediately see the link to a result that we previously accepted. I ask how many drew a diagram, labelled angles, drew extra lines, wrote calculations as examples of techniques that can be helpful or necessary in finding a proof and demonstrating the link to the earlier result. The final group task is to decide which geometric results we would accept as true and not needing further justification. I obviously need to control this list for the purpose of the next work so I will act as an informed mathematician commenting on their choice, asking them to choose between similar definitions (e.g. 180 degrees in a straight line or 360 degrees round a point) and completing the list to include similar triangles, well-defined triangles and (two of) alternate, corresponding and opposite angle theorems.

The tasks that the students then attempted were: proving that the sum of the interior angles in a triangle is 180 degrees, that dropping a perpendicular from the centre of a circle to any chord bisects that chord, that in an inscribed triangle ABC the angle between the chord AB and the tangent to the circle at A is equal to the angle at C, investigating the sum of the angles at the points of any star, and the shape formed by joining the midpoints of the sides of a quadrilateral. Most preferred to try different tasks rather than concentrate on finishing each one, although no one wanted to leave the session without proving the angle sum in a triangle. Almost everyone used Cabri for each task even if just to illustrate. My role was to talk to individuals about their strategies. The technique that became the recurring feature of these conversations was adding construction lines, whether on paper or on the screen, and the most obvious contribution of Cabri was in drawing a tangent. This construction necessitated finding the centre of a circle and putting in an extra radius, which was then useful in defining new angles and using known results. After doing this task students seemed happier about not exactly reproducing the drawing in other tasks. The least helpful use was in constructing the stars - students were unwilling to spoil the diagram on the screen.

Discussion and Conclusions

I expected three kinds of learning to be happening simultaneously during these activities, each needing its own evaluation. All students learnt some mathematics by solving geometry problems which were not immediately obvious, and I have discussed above some of the ways in which using Cabri

techniques helped them to a solution. In learning *about* mathematics I would include the ability to reflect on progress in a task. Compared with their work in other sessions I was surprised at how confident the students seemed in trying out different approaches. Some of these approaches were basic experiments, such as dragging points to see what happened, but the students readily recognised this. Other students had screens covered with figures but explained that at present they were working with this part, and would go back and fix another bit later. Pointing to the display and describing actions helped them to explain to me and to themselves what they were trying to do. The experience of using Cabri did seem to influence what sort of strategies were used. One student argued with her friend against using Pythagoras's Theorem to prove a simple result because it wasn't "the right sort of theorem … not on the computer", getting a glimpse of a deductive structure of geometry. In making conjectures, using Cabri shifted the emphasis from finding the 'right' thing to notice - everybody could see something happening - to how to describe what they saw and comparing different conjectures. The third kind - learning about learning - is still at the exploratory phase. The students will use their Cabri experiences amongst others for future reflective writing in a concept identification phase. One particular comment confirmed for me that this learning was taking place - when a student looked around thoughtfully and said " Have you noticed? We start each session sitting with our friends and by the end everyone has gone off to separate computers to work in their own way."

References

Bellemain, F. and Capponi, B. (1992) Specificities of the organization of a teaching sequence using the computer *Educational Studies in Mathematics*, 23, 1, pp.59-97 (in French).

Clements, D. and Battista, M. (1992) Geometry and spatial reasoning in Grouws, D. (ed.) *Handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics*, New York, Macmillan, pp.420-464.

Gagatsis, A. and Patronis, T. (1990) Using geometrical models in a process of reflective thinking in learning and teaching mathematics *Educational Studies in Mathematics*, 21, pp.29-54.

Hanna, G. (1995) Challenges to the importance of proof *For the Learning of Mathematics*, 15(3), 42-49.

Hasegawa, J. (1997) Concept formation of triangles and quadrilaterals in the second grade *Educational Studies in Mathematics*, 32(2), 157-179.

Hazzan, O. and Goldenberg, E.P. (1997) An expression of the idea of successive refinement in dynamic geometry environments *Proceedings of the 21st conference of the International group for the Psychology of Mathematics Education*, 3, 49-56.

Healy, L., Hoelzl, R., Hoyles, C. and Noss, R. (1994) Messing up *Micromath* 10, 1, pp.14-16.

Hoyles, C. (1995) Exploratory software, exploratory cultures? in DiSessa, A., Hoyles, C. and Noss, R. (eds.) *Computers and exploratory learning*, Berlin: Springer-Verlag, pp.199-220.

Hoyles, C. (1996) Modelling geometric knowledge: the case of the student in Laborde, J.M. (ed.) *Intelligent learning environments: the case of geometry*, Berlin, Springer-Verlag, pp. 94-112.

Laborde, J.M. (1996) Intelligent microworlds and learning environments in Laborde, J.M. (ed.) *Intelligent learning environments: the case of geometry*, Berlin: Springer-Verlag, pp. 113-132.

Noss, R., Healy, L. and Hoyles, C. (1997) The construction of mathematical meanings: Connecting the visual with the symbolic *Educational Studies in Mathematics*, 33, 2, pp.203-233.

Noss, R., and Hoyles, C. (1996) *Windows on mathematical meanings: learning cultures and computers*, Dordrecht: Kluwer.

Perks, P. and Prestage,S. (1999) Square roots - an algorithm to challenge*, Mathematics Education Review*, 10, pp.31-40

Rogers, P. and Kaiser, G. (eds.) (1995) *Equity in mathematics education: influences of feminism and culture*, London: Falmer.

Schoenfeld, A. (1992) Learning to think mathematically: problem solving, metacognition and sense making in mathematics in Grouws, D. (ed.) *Handbook of research on mathematics teaching and learning*, New York: Macmillan, pp.335-370.

Simon, M. (1994) Learning Mathematics and Learning to Teach: Learning Cycles in Mathematics Teacher Education *Educational Studies in Mathematics*, 26, 1,pp.71-94

Wilensky, U. (1991) Abstract meditations on the concrete and concrete implications for mathematics education in Harel, I. and Papert, S. (eds.) *Constructionism,* Norwood, New Jersey: Ablex Publishing Corporation, pp. 193-203.

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A Proving Situation: Do we rise to the occasion?

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In previous years one of the sessions we run with our pre-service secondary mathematics teachers has considered the nature of proof with the major focus being on their learner knowledge. In this latest session an activity designed to enhance pedagogical content knowledge was included. This paper describes the activities and offers analysis of the effectiveness of the session.

A session on proof has been a regular feature in our secondary PGCE course. The session reflects a number of principles in our teaching defined in an earlier paper, *Square Roots* (Perks and Prestage 1999). We aim to challenge existing beliefs held by students, beliefs about the nature of mathematics, the nature of their learning and how others learn. Add to this also our belief that student teachers need to re-think their own subject knowledge and transform it in some way before using it in the classroom. When constructing sessions we engage the emotions, work on students' own experiences and, in this case, we asked the students to report on their own solutions in order to raise issues relating to teaching ready for further analysis.
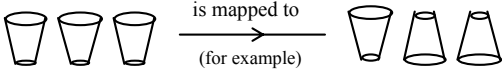
Our reasons offering a proof session have always seemed clear (Perks and Prestage, 1995). Proof is one of those areas where the students' mathematical subject knowledge is limited, a fact supported in a piece of writing on the session by one of our good mathematics graduates:

> *I was very glad to be looking at proof, since it is part of mathematics which I find difficult and puzzling. I find it difficult because of how little I have been taught about it. The first time I really encountered it was at university, where we were shown various ways of proving mathematical statements, but I never learned how to come up with a proof, i.e. where to start from.* (Lucy, June 1999)

Lucy is typical, the learner-knowledge of these graduate mathematicians is that proof is hard, something done at university, generally algebraic and not part of the 11-16 curriculum.

The content for the session was five questions, each offered in turn to the students in order to preserve the focus of each question. A brief period of time

was allowed for each question so that the students would work individually on the solution. Our reasons for the choice of each question is explained below.

1. Prove that the sum of three consecutive numbers is a multiple of three.	*Here is the starter for 10, a nice comfortable algebraic solution is expected from the students which is what they produce!*
2. Prove that the product of three consecutive numbers is a multiple of six.	*It sounds as if another algebraic solution is required but the students soon become stuck. The proof is a wordy explanation of the properties of 3 consecutive numbers, and requires the students to break set. Alternative methods are required.*
3. Prove that all prime numbers after 2 and 3 are of the form 6n ± 1, given that n is a positive integer.	*Immediate consternation! No-one has proved anything about prime numbers*
4. Prove that n^3 - n is a multiple of six, for n being a positive integer.	*Phew! Back to algebra, but ... the algebra does not help beyond the factorisation.*
5. Given 3 tumblers, all face down, you can turn over exactly and only two at a time, is it possible to have a situation where all are upside down?	*Here is a complete contrast; the proof appears to be by exhaustion but how do you know when to stop?*

In the summer of 1999 we included some items from the Justifying and Proving in School Mathematics project (Healy and Hoyles, 1998, Hoyles and Healy, 1999). We chose to use the material (Hoyles and Healey, 1999, p.21) as a focus for students commenting on the pupils' solutions to consider more explicitly proof in the secondary curriculum, to share alternative approaches in a wider form than the original session and crediting pupils with rational ideas, even if these are not transparent. The students worked on these items before answering the five questions above.

The session began with this commenting on the pupils' proofs and the students were told that their writing on these proofs would be photocopied for us to keep

and later analyse. The students were asked to work on their own and to write down what they would put on the pupils' scripts to help them. They were given thirty minutes to write, without conferring. *(Working individually is an important aspect of a session, not to test but for the students to examine their own knowledge and beliefs.)* The distinction between marking and assessment emerged as an important issue from this part of the session but we will leave that commentary for another paper.

After the marking the session continued as in the past with the students working on their own on the five questions. The first question, as always had most of the students scribbling, the second saw them begin with confidence, but then, in most cases become puzzled. The third question was met with the usual disbelief and a welcome sigh greeted the fourth - algebra is so reassuring. As for the fifth, was it true? There was nothing different from the sessions in earlier years.

Discussion of the students' own solutions followed. The presentation of the argument for question one convinced everyone in the room. The only differences in method seemed to be the choice of letter, n being the most popular, followed by x, and the use of n-1, n, n+1 or n, n+1, n+2. The discussion did move quickly to the words which were lacking in the presentation on the board and this led to a comparison between the student-teacher's work on the board and Arthur's proof. This in turn led to a re-scrutiny of the proof and a consensus that it did need some more words to help the acceptance of definitions. *The students had started to work on their own learner-knowledge*, in particular the nature of definitions. Pat then offered an image (figure 1) for proving that two odd numbers make an even number using an anecdote from a 7 year-old who was explaining his solution:

… because odd numbers always have this sort of pattern on the number boards …
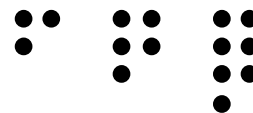
… then two numbers could be joined to form something similar to the even number boards
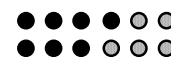
figure 1

A 7 year-old proving? Is this really proof? This provokes lots of challenge to the students' learner-knowledge, particularly the reaction, "Isn't proof hard?"

The teacher did not act convinced and the child was forced into a more explicit extension into generality. When does a specific assume a generality? How does the teacher know what to say? How many examples convince? Here we are also working on the students' practical wisdom. The pupil did this by

covering parts of the dots with his hands, as the clouds in figure 2, to hide how many pairs of dots were unseen.
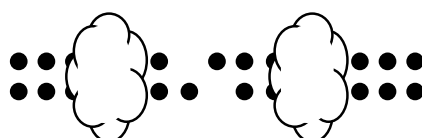

figure 2

Could a proof of question 1 be expected from all 11 year-olds? What would be an alternative proof using imagery? Build three towers of cubes, see figure 3, using differently coloured cubes for the add 1 and add 2 and the general can be seen in the particular. Like the 7 year-old the generality can be made more explicit by the use of hands to hide some cubes and thus remove the special number property from the example.
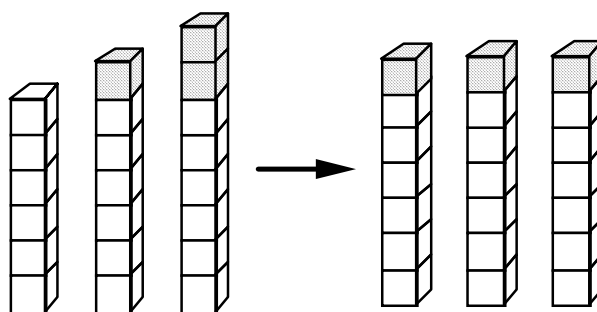
figure 3

The students were amazed. One student commented that she felt that the image was really strong. Another commented that *she* could now see n, n+1, n+2 and the related 3 lots of n+1. The learner-knowledge for an algebraic type of proof seemed secure; it was the students' first choice and algebra was straightforward. As was our intention the pictures challenged some students to consider the nature of proof.

In discussing the first question and its solution many, many issues arose. *We should not be surprised that the complexity of transforming subject knowledge for the classroom, pedagogical content knowledge (Shulman, 1986), baffles some of the students*. In summary the pedagogical content knowledge (pck) addressed included issues such as:

- the written form of the proof on the board needs. to match the language of the spoken explanation;
- the role of definitions in axiomatic proof;
- different forms of proof may be accessible to different learners;
- the variety of ways of moving from special cases to generality.

Like good teachers we know that one example is insufficient to extract a generality so we move onto the next question to raise similar issues in a different context. For the second question, most had begun with algebra, a few had continued to a proof in words but were reluctant to share their solution and no-one offered to write on the board. Most could just remember the term proof by exhaustion but remained unconvinced by the arguments until some work on definitions and the structure of the number system had been explored. This learner-knowledge was accompanied by the rehearsal and clarification of related language and definitions. The pck considered included:

- awareness of language and ambiguity in definitions and symbols;

- knowledge about number facts is not necessarily transferable;

- the explainer being convinced does not necessarily make the hearer convinced, despite repetition;

- alternative strategies are necessary in explanations;

- The obvious is not always obvious.

For the third question, an examination of leaner-knowledge about primes includes the half-remembered fact that no formulae are available to generate prime numbers which supports the belief that this proof is difficult. *The understanding of the task has to be challenged*. The pck included discussion of the use of Sieve of Eratosthenes to generate primes and the presentation of this method, especially when the shading of multiples is done on a numbers grid which is six wide.

The fourth question, realised to be similar to the first, led to a factorisation of the expression but not much more. The next step remained elusive for many.

In the discussion of the proof for the last question, no-one would admit to anything written, but verbal explanations were offered and eventually a state-map was drawn by Pat in response to their statements, much to their relief. (No-one offered the group combination table.)
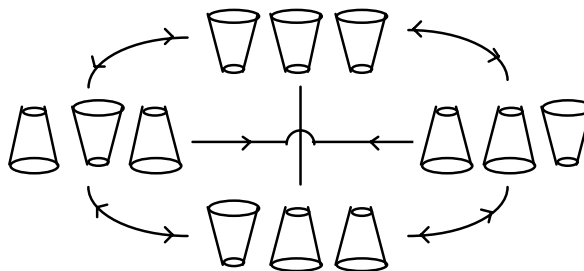

figure 4

The picture appears to offer something which convinces and is less surprising than the earlier image using the cubes, possibly because the cups were real

objects which are not normally represented by letters, so that the picture could be considered an abstraction, whereas adding pictures to number may be conflicting with the transition from object to abstraction normally encountered in the development of number and algebra.

The session went well, the students commented on how useful it was, so why are we dissatisfied? In terms of the objectives, identified at the beginning of this paper, the session had been a success. Some subject knowledge had been worked on; some practised; some met for the first time; some misconceptions had been challenged; some alternative proofs had been presented and discussed as to their conviction. Some pck had been addressed to complement what had been done in previous sessions and learned on their main teaching experience in the spring term. The students had had the opportunity to practise their subject knowledge in the context of proof and their pck in terms of writing on pupils' work and the presentation of this aspect of mathematics. However, at the end of the session the students returned to the examples of written proofs from the pupils (Hoyles and Healey, 1999, p.21), and in groups were asked to write new comments. It was our analysis of this writing, which failed to add much more explicit written detail, that provoked us to reconsider the session. We have expectations of the session which may not transform into knowledge for our students.

- The session is linked within the framework established across the course as a whole, where any session has to be considered on three levels, the student as learner, the student as future teacher and the tutors as teachers. Should aspects be made more explicit and how?

- The work on subject knowledge remained at the level of learner-knowledge, i.e. the knowledge needed to answer the question. The modelling of the solutions offered examples of classroom events. There was no attempt to synthesise the learner-knowledge and the pck to extend the knowledge of the mathematics which can be made explicit, i.e. teacher-knowledge.

- How might this session be linked to the role of generalisation within the rest of the curriculum, the formulae for the area of a rectangle, the circumference of a circle, the sum of the angles of a triangle for example? For some of our students this is likely to happen (despite any explicit input from us, as in all teaching).

With mathematical proof, I think you can prove things that you do not know, and know things you cannot prove. An example of the first case is ..."The sum of three consecutive numbers is a multiple of three." I did not know this was true, but had an idea of how to begin proving it ... I think the other case occurs often in mathematics, for example, we know

the angle sum of a triangle is 180°, but most people do not know how to prove it.   *(Lucy, June 1999)*

- Our concern is how this link might be established more explicitly in this session, despite its consideration elsewhere. How might working on proof be related more effectively to the students learning how to explain (or to allow the situation to explain) the mathematics of such generalisations?

- As with any session we offered examples which have to be, by the nature of the task, special cases. How is this to be translated into teaching? If exemplars are not absorbed as generic examples they remain as specific; I may only remember this session when I want to teach about odd and even or the other specific questions. We work through reflection, and for a number of our students this is sufficient for them to see the session in terms of its representation of many others. This is not true for all students, so how might you work more explicitly on the generality this is intended to represent?

What routines, questions, patterns of activities could we in initial teacher education use to help synthesise the learner-knowledge with the developing pck to produce a level of subject knowledge which recognises the generality and the progression of the content (teacher-knowledge) (c.f. Prestage and Perks, 1999).

References

Healy, L. and Hoyles, C. (1998) *Technical Report on the Nationwide Survey: Justifying and Proving in School Mathematics*, Institute of Education, University of London.

Hoyles, C. and Healy, L. (1999) Students' views of proof*, Mathematics in School*, 28, 3, pp.19-21.

Perks, P. and Prestage, S. (1995) Why don't they prove*?, Mathematics in School*, 24, 3, pp.43-45.

Perks, P. and Prestage, S. (1999) Square roots - an algorithm to challenge, *Mathematics Education Review*, 10, pp.31-40.

Prestage, S. and Perks, P. (1999) *Subject matter knowledge in experienced and novice teachers of mathematics*, paper presented at the annual conference of the British Educational Research Association, Brighton:
http://leeds.ac.uk/educol/documents/000001168.htm

Shulman, L.S. (1986) Those who understand knowledge growth in teaching, *Educational Researcher*, 15, 2, pp.4-14.

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Working on wonder and wondering: making sense of the spiritual in mathematics teaching

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In this article two sessions with PGCE Secondary Mathematics students are described. The aim of the sessions was to explore what could be meant by references in the Standards for Qualified Teacher Status to personal, spiritual, moral, social and cultural development. In particular, the relationship between these aspects of adolescent life and the teaching and learning of mathematics were discussed.

Introduction: the Standards

The following two phrases are contained in the Standards for Qualified Teacher Status (DfEE, 1998):

> *Plan opportunities to contribute to pupils' personal, spiritual, moral, social and cultural development.*

> *Use teaching methods which sustain the momentum of pupils' work and keep all pupils engaged* **through** *exploiting opportunities to contribute to the quality of pupils' wider educational development, including their personal, spiritual, moral, social and cultural development (my emphasis).*

These passages seem to be referring to a two-way process: in the first case working on pupils' wider development through mathematics; in the second, working on mathematics through the pupils' personal development. In either case there are difficulties understanding what was envisaged by the authors and in making sense of a relationship between mathematics and the five listed aspects of development. However, an attitude of cynicism about their meaning does not seem to be a fully appropriate framework for working with students on the issues, given that at some time in their futures they may be faced with the requirement to work within similarly obscure guidelines.

Using the Standards in sessions

I decided to spend two mornings working on these phrases with Secondary PGCE mathematics students. The timing was generous if one considers the

status of the phrases in the ITT curriculum, but working on meanings does take time and I wanted to allow plenty of space to explore what they might mean in terms of the secondary curriculum.

The sessions took place at the end of the second term, during which students had spent nearly all their time in school.   I had been disappointed by some of the lessons I had seen in school in the weeks before these sessions.   Students who had started the course with imaginative ideas, developed interactive approaches and been quite skilled in using pupils' ideas had fallen into routines of covering the syllabus, using words like push, drive and falling behind to describe their teaching. I wanted to see if their earlier visions of themselves as teachers could be reawakened. If there are meaningful and motivational links between mathematics and personal, spiritual, moral, social and cultural development for pupils, then perhaps I could use these same links to also contribute to the students' development.

The sessions were therefore multipurpose. At a superficial level they were intended to fulfil overtly a requirement for *accountability*; at a professional level to provide one *model of working* with such requirements; at a pedagogic level to explore the *meaning of teaching* through and towards these aspects of human development for pupils; and at the level of *students as learners* to work through and towards the same aspects. Whether these levels are clear at every stage of the sessions I leave to the students, and the reader, to judge; I introduced them at the start, but this does not mean that the students were thinking about them at every stage. What is more likely is that they brought with them their own perspectives on the course, their progress and the discursive practices of their placement schools and the need to reflect with peers on the term's teaching. Each student will have had a position relative to these and, in the session, will have worked on these within the framework I offered.

Firstly I shall outline the principle basis for the approach I adopted, then describe the sessions and, as far as I can, their outcomes.   The principles which underlie my practice as a teacher-educator are belief-based rather than research-based, although in places they accord with much research literature, particularly in relation to school-situated learning (Lave,1988; McIntyre and Hagger, 1996; Leinhardt, 1988), reflective practice (Schon, 1987; Korthagen, 1988), students' construction of knowledge about teaching (Eurat, 1994; McIntyre, 1988; Calderhead, 1988) and personal change (Eraut, 1994).

They are:

- learning to teach is a process of personal change, involving growth of awareness and change of belief;

- learning in school is mainly achieved through induction into normal practice by observation of ways of behaving and ways of discussing issues;
- progress in teaching is achieved through identifying problems and applying cycles of problem-solving and evaluation;
- students construct their knowledge of teaching through their experience, which is observed and interpreted according to their current state of belief and knowledge;
- without intervention students will problematize their teaching only according to their own belief frameworks;
- one of the roles of the HE institution is to suggest other ways to look at practice, to initiate structures of reflection, to create a forum for examination of specific issues, to air multiple possibilities (including the ways we teach students);
- students are sometimes self-centred, sometimes pupil-centred, and sometimes syllabus-centred and need space to integrate these standpoints and others;
- learners should not be placed, by the decisions of their teachers or authorities, in positions where they have to openly challenge existing practices;
- learning about external demands on teachers is part of becoming a member of a profession.

The sessions

Bearing in mind the need to make sense of the diverse demands made of mathematics teachers, I started with a series of questions:

What are the roles and purposes of education?

After small-group discussion two contrasting views, each centred on individuals, emerged:

1. Education is to fit the pupil for future economic life, employment, dealing with finances, political decisions etc.
2. Education is to develop the pupil, the whole person, to become well-rounded and develop their talents or potential.

Then why do governments pay for education?

This second question prompted students to mention the role of education as producer or reproducer of future kinds of society; as producer of a certain kind

of workforce rather than individual employment; as developer of informed citizenship rather than merely understanding individual needs.

What *is the role of the teacher in relation to these? And the mathematics teacher in particular?*

Since the students were more inclined to think about the education of individuals, this third question gave encouragement to think about the relationshp between the personal development of each adolescent and how this may relate to social ideals. They responded with statements about teaching mathematics which would be useful, contextualising mathematics so pupils could see how it was useful, the idea of teaching pupils in school what they would then use outside, and pastoral role of teachers. One student added that the atmosphere one created in the classroom also had a role to play: mutual respect, teamwork, listening, sharing and teacher intervention in social situations were classroom features which, to some extent, could aid pupils' personal development.

Within the group there was argument: "When did you ever have to solve a quadratic equation?" was asked; one response was that mathematics helped you think, trained the mind and so on. Others thought that the mind could not be effectively trained in three lessons a week.

Within the first few minutes of the session, therefore, some of the common responses to the phrases in the Standards had been aired and examined through consideration of a philosophical question using their collaborative knowledge. These things having been said, and discussed among them, we could move on to more challenging ground. I had a sense of a common knowledge about the approaches mentioned so far, possibly gained in school and through reading. This part of the session demonstrated the importance of finding out what students know already, so that subsequent work can take into account their current views. We were obviously going to have to work hard to get beyond pastoral, behavioural and utilitarian interpretations, valuable though these are. I include myself in this work because the development of a coherent view of mathematics pedagogy which includes these aspects of the Standards had so far eluded me, and these sessions were as much a chance for me to work on the issues as the students.

Shared evaluation of resources

All the schools in which our students learn to teach are well-resourced, so the process of evaluating resources seemed a familiar way to develop ideas and

see what general features emerged when the five features, plus mathematical content, were used as analytic categories.

First they read an article from Mathematics Teaching (Downes, 1997) and evaluated its contents and the accompanying resources in terms of the five aspects and mathematics[1]. The aim was to gain a critical understanding of Downes's interpretation of some of the issues. There was then a discussion to share evaluations based on different perspectives. My aim was to help them see that people approached the task from different positions and different prior knowledge and thus, I hoped, they would broaden their own perspectives a little. The article was about a maths week in a school which used the requirement for collective worship as a framework within which pupils considered mathematical questions arising from contexts such as rainforest destruction, malnutrition, trading inequities and clean water. Most students found this interesting and could imagine setting up something similar themselves, although one was cynical about whether adolescents would be interested in such issues. The mathematics, however, was seen to be limited to calculations of varying complexity and data-handling, and the whole activity was a specially-structured piece of work, not seen as part of the usual curriculum. Students reported (as their mentors had during the planning stage) similar special one-off mathematics activities in their schools: one had an African week during which Egyptian numbers and pyramids had been the mathematical topics; another had used a complex trading game (Oxfam) to model multi-national trading inequities.

Having shared this task students were then asked to evaluate a variety of resources for teaching mathematics, again using the five aspects and mathematics. Their evaluations, and the reporting back on their findings, took most of the time available. I felt that it was important to give time to work on the mathematics suggested in the resources and to think how it might fit into a normal school curriculum[2]. I will not use space here to report their conclusions, merely to say that as well as the common responses described in relation to the Downes article, students were mainly finding what they believed to be motivating contexts within which core curriculum mathematics could be approached. Some of these were historical, others social, all in some way harnessed normal concerns of adolescents finding their place in the world, testing out their beliefs and attitudes, and trying to become adults. The contexts themselves were not necessarily mathematical, nor were they usually treated mathematically in their natural occurrence, but did provide raw material

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[1] I am grateful to Linda Haggarty, University of Reading, for directing me to this resource

[2] I have listed the resources at the end of this article.

for raising mathematical questions, modelling and otherwise mathematising. For example, mathematical thinking may not normally be applied to textile design, although a mathematician might see mathematical structure within the finished result; the inherent structure, however, allows for mathematising in the classroom.

The students had clearly grasped that one way to attract pupils' attention to mathematics was to show them how it related to their current interests, not just things they might need later or situations outside school which may not be familiar to them. The alienation created for some by using contexts which might, nevertheless, motivate others (such as using football as a context) was something they articulated clearly. Most students were also aware of gender and racial stereotyping in textbooks and some mentioned the tokenism which can result from a well-meaning attempt to counteract these, such as using Asian names and poppadums in place of European names and pizzas. Prescribed reading earlier in the course (Shan and Bailey, 1991; Scott-Hodgetts, 1986; Dowling, 1994) and their discussions with mentors, as well as previous experience, had prepared them to notice these relatively common responses to the personal, spiritual, moral, social and cultural aspects of development.

Supporting students' views

Since the students' sense of adolescence as a particular phase of development, with its own concerns and preoccupations, appeared to be quite strong I used passages from *Mathematics Curriculum 5 to 16* (HMSO, 1985) to give shape and language to this sense, in relation to mathematics teaching. This also served to show (a) that their sensitivity was worthy and important and (b) that authority in education need not always speak with the current discourse of targets, levels and standardisation. The following will give a flavour:

> *There is a fascination about mathematics itself ... which it is possible to develop to some degree in most if not all pupils. This fascination will not, of course, be the same for all pupils but most aspects, if considered within a suitable context and at an appropriate level, can have such appeal.(p.3)*

> *The aim should be to show mathematics as a process, as a creative activity in which pupils become fully involved, and not as an imposed body of knowledge immune to any change or development. (p.4)*

It was important to spend time on authoritative support of students' sensitivities, because when learners find themselves voicing opinions which contradict current orthodoxy (and in some cases had left students in conflict between the way they wanted to teach and the pressures of covering syllabi)

they may need more than the reassurance of a teacher or tutor to believe their views are worthwhile.

In summary so far the students and I had discussed several issues of cultural and contextual specificity of mathematics which arose from the resources:

- mathematics as a route to employment;
- mathematics as a good way to think;
- the value of special activities in school;
- the limited nature of the mathematics arising from some socially and ethically valuable considerations;
- the pastoral role of the teacher;
- how classroom norms could model socially acceptable behaviour;
- the role of history and culture in generating interest in mathematics;
- awareness of gender and racial bias in textbooks and the dangers of tokenism; harnessing adolescent concerns in mathematics teaching.

Spirituality

However, there was an element of the five aspects which had so far not been discussed, that of spirituality. I was unwilling to leave this at the level of Downes's article, using collective worship as a framework for dealing with socially-responsible mathematics, nor to leave it to those who had randomly dipped into the Charis materials provided for evaluation (1996, see resources list). The idea that spirituality could link mathematics with a liberal Christian concern with charity and social justice did not appear to do justice to the human sensitivities which might fundamentally drive us to seek for explanations of the world in terms of religion, philosophy, science and mathematics. I wished to link spirituality with our propensity to regard the world with wonder, and to follow this with wondering.

I had used this description of spirituality earlier in the course when I asked the students what aspects of mathematics filled them with awe and wonder. A few said infinity and we left it at that. I had also given them an article by Movshovits-Hadar (1988) in which she writes about using surprise in her teaching: for example, introducing the theorem of Pythagoras as a surprising special case of adding the squares on the sides of general triangles. Some had regarded this to be "a little far-fetched" in terms of a practical teaching approach.

Wonder and wondering

I lowered the lighting in the room, said to them that I was going to read to them about wonder and asked them to settle into comfortable listening positions. In itself this was a different and, I expected, an intriguing approach. I read:

> *To characterize wonder we are forced to look at its alternative, the qualities of the ordinary, and paradoxically what we end up saying is that there cannot be any experience of the ordinary. As a result, surprise, the eliciting of notice, becomes the very heart of what it means to have an experience at all. ... The ordinary can not or does not turn itself into experiences. (Fisher, 1998, p.20)*

I asked them to consider privately what, if this were true, this might mean for the teaching of mathematics and whether ordinary lessons and ordinary work would encourage learning. I read on:

> *The tie between wonder and learning is clear in the moment when after long confusion and study you suddenly say, Now I get it! Plato ... uses mathematics because the moment of getting it is extremely clear in mathematics. (p.21)*

> *The passage from wonder to thought sets off a chain of experience built on ever repeated, small-scale repetitions of the experience of wonder. The first global moment of wonder is relocated, or better yet, reactivated, kept alive at every step within the process of thought itself. It is not the stimulus to thought, but the very core of energy that makes up each moment of thought. (p.41)*

Again I paused for them to think about what that might mean for mathematics lessons, and of the meaning and purpose of step-by-step approaches. Steps need not be devised which smooth the path (Wigley, 1992) to a solution. Indeed, such small steps may not result in learning at all. Instead, steps could be devised which offered opportunities for wonder and wondering.

I then referred back to one of the activities they had evaluated earlier:

Mark a point P inside a circle of radius 8 cm. fairly close to the edge. Fold and crease the circle so that the circumference just touches the point, draw the crease in pencil. Repeat the folding and drawing procedure several times. An ellipse appears.

When the first student had tackled this, her cries of delight had attracted others to it. Before long a group of several students had surrounded the work and began to explore it for themselves; some worked on paper, others had gone

away to draw it using Cabri; there were several conjectures around about the position of the foci and directrices of the ellipse. I was aware that some had worked on it between the two sessions. One student had described it as wonderful. In this case, I suggested, wonder had obviously been an effective motivator for people to work in a sustained way on the situation, raise new questions, look for ways to explain their insights to others, and even attempt to communicate their enthusiasm to others. By referring directly to their own shared earlier experience, which I named as wonder, I hoped to bring them out of quiet reverie and into an atmosphere of discussion.

I asked students to write down what aspects of the secondary mathematics curriculum they *now* felt they could teach through generating a sense of wonder among pupils. Of course, it was accepted that not every pupil would respond similarly; nevertheless it was appropriate to construct situations which *might* engender wonder.

Support from authority, which I highlighted as important, comes again from HMSO (1985):

> *The spark may come from a feeling for order, the appreciation of pattern, an interesting relationship, the power of a formula, the simplicity of a generalisation, a curious or unexpected result, the conciseness of an abstraction, the aesthetic appeal of mathematical designs or models in two or three dimensions, or the elegance of a proof. (HMSO, p.4)*

The list the students produced was wide-ranging and exciting; far more than the infinity mentioned earlier in the course:

> pi, sin2 + cos2 = 1, circles, circle theorems, internal and external angles of polygons, angles, graphs, trigonometry, sum of arithmetic series, iterative solutions, sequences, probability theorems, the uncertainty of probability, difference between two squares, tessellations, binomial expansion, symmetry, theorem of Pythagoras.

Conclusion

In some ways I can say the sessions I have described above were a success: a range of approaches relating to the personal, spiritual, moral, social and cultural development of pupils had been discussed; they had been introduced to several useful resources; their hunches about good ways to work with adolescents had been affirmed; we had, relatively fearlessly, worked on harnessing a spiritual dimension through wonder, including their own, and turning it into wondering.

The list of mathematical contexts for wonder, and hence for spirituality, indicated that their understanding of the possibilities of teaching had expanded hugely.

The real test of success is whether they carry this into their teaching, or whether, like their earlier visions and bravely interactive lessons, their sense of wonder becomes sidelined once again by the discourse of coverage.

Resource List

Bell, R. and Cornelius, M. (1988) *Board Games from Around the World,* Cambridge: Cambridge University Press.

Howse, J. (1976) *Maths or Magic: simple vedic arithmetic methods* , London: Watkins.

Nelson, D., Joseph, G.G., and Williams, J. (1993) *Multicultural Mathematics,* Oxford: Oxford University Press.

Nock, D., Riding, M. and White, N. (1988) *Mathematics for All* , Trowbridge: Wiltshire Education Authority.

Shan, S-J. and Bailey, P. (1991) *Multiple Factors*, Stoke-on Trent:Trentham Books.

Shortt, J. et al. (1996) *Charis mathematics*, *Units 1to 19*, St. Albans: Association of Christian Teachers.

V & A Education (1996) *Teaching Maths through Islamic Art,* London:Victoria and Albert Museum.

Zaslavsky, C. (1973) *Africa Counts*, New York: Lawrence Hill Books.

Bibliography

Calderhead, J. (1998) The Development of Knowledge Structures in Learning to Teach in J. Calderhead (ed.) *Teachers Professional Learning,* London: Falmer.

DfEE (1998) *Teaching: High Status, High Standards*, Requirements for Courses of Initial Teacher Training, Circular Number 4/98, London: DfEE.

Dowling, P. (1991) A touch of class: ability, social class and intertext in SMP 11-16, in D. Pimm and E. Love (eds.) *Teaching and Learning School Mathematics*, London: Hodder and Stoughton

Downes, S. (1997) Mathematics and Collective Worship, *Mathematics Teaching,* 159.

Eraut, M. (1994) *Developing Professional Knowledge and Competence,* London: Falmer.

Fisher, P. (1998) *Wonder, the Rainbow and the Aesthetics of Rare Experiences,* Cambridge, Mass.: Harvard University Press.

Lave, J. (1988) *Cognition and Practice,* Cambridge: Cambridge University Press.

HMSO (1985) *Mathematics from 5 to 16*, London: Her Majesty's Stationery Office.

Korthagen, F.A.J. (1988) The Influence of Learning Orientations on the Development of Reflective Teaching in J. Calderhead (ed.) *Teachers Professional Learning,* London: Falmer.

Leinhardt, G. (1988) Situated Knowledge and Expertise in Teaching in J. Calderhead (ed.) *Teachers Professional Learning,* London: Falmer.

McIntyre, D. (1988) Designing a Teacher Education Curriculum from Research and Theory on Teacher Knowledge in J. Calderhead (ed.) *Teachers Professional Learning,* London: Falmer.

McIntyre, D. and Hagger, H. (1996) *Mentors in Schools: developing the profession of teaching,* London: Fulton.

Moshovitz-Hadar, N. (1988) School Mathematics Theorems – an endless source of surprise, *For the Learning of Mathematics,* 8, 3, pp.34-40.

Schon. D. (1987) *Educating the Reflective Practitioner,* London: Jossey-Bass.

Scott-Hodgetts, R. (1986) Girls and Mathematics: the negative implications of success in L. Burton (ed.) Girls into Maths Can Go. London: Holt, Rinehart and Winston.

Shan, S. and Bailey, P. (1991) *Multiple Factors: classroom mathematics for equality and justice.* Stoke: Trentham Books.

Wigley, A. (1992) Models for Teaching Mathematics, *Mathematics Teaching,* 141, pp. 4-7.

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The Use of Images in Teacher Education

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The intentions of this article are to briefly rationalise the role of imagery in the teaching of student teachers, to exemplify such images and to invite readers to share images which they use in their own work with student teachers.

Theory

The development of using images in teacher education was brought about by the apparent lack of impact that traditional methods of teacher education appear to have had upon student teachers' preconceptions (e.g. Kagan, 1992). Much of the theoretical background to the use of imagery in teacher education is summarised in an excellent paper "Taking Account of Student Teachers' Preconceptions", by Theo Wubbels (1992). Wubbels develops the argument, originating with Watzlawick (1978)

> *... that experiences and knowledge about everyday life result in images that represent people's constructions of reality. These images are the essential knowledge of people about the world: world images. (Wubbels, 1992, p.139)*

It is thought that world images are processed in the right hemisphere of the brain and that they are difficult to reach and change using logical, analytic, rational approaches, which are processed in the left hemisphere of the brain.

> *If logical, left hemisphere language is used to reach the world images, the language of images has to be translated into the language of explanation, argument, analysis, confrontation, interpretation, and so forth. When student teachers are approached with this language they might feel invited to engage in a process of rationalising their images rather than changing them: in this way the left hemisphere could act as a guardian to keep the right hemisphere images unchanged. (Ibid. p.139)*

Wubbels goes on to offer several teacher-educator interventions that might help to build or rebuild world images through 'right hemisphere strategies'. In this article I focus on one of these strategies - the use of metaphors, models, analogies and the presentation of alternative images.

Evidence exists that world images *can* be changed and that this is worthwhile for teacher education. Joram and Gabriele's (1998) study suggests that targeting student teachers' imagery has a significant impact in changing preconceptions.

> *We suggest that teacher-educators invite preservice teachers to entertain other alternate realities, work with powerful analogies and metaphors, and consider how new ideas are compatible with, and augment their existing beliefs. (Joram and Gabriele, 1998, p.178)*

Evaluating the success of applying such a strategy on a course in educational psychology in comparison with conventional approaches, Joram and Gabriele claim considerable success with responses to three different questions as follows:

> *Only 8% of the preservice teachers felt their views of learning and teaching had not changed as a result of taking the course …*

> *49% of preservice teachers felt that a significant change had occurred in their views of learning …*

> *57% of students felt their views of teaching had undergone a significant change. (Ibid. p.184)*

Examples from my own teaching

I use an image of learning to play a violin to try to help students see, among other things, that the "apprenticeship of observation" (Lortie, 1975) is not adequate, and may in fact be misleading. Teaching is not unproblematic.

> *I have* seen *people playing a violin and it **looks** really easy. I **observe** the bow scraping the strings, the way to hold the violin and the pensive, concentrated facial expression. Perhaps it really is that simple, I am unconscious of my own incompetence in this area.*

> *Once I start to play the violin I suspect that my apprenticeship of observation and imitation is not going to be sufficient in practice. I become conscious of my incompetence. I get some tuition and painstakingly slowly begin to gain some skill with the violin, if I am very careful I can play some simple tunes, I've reached the level of conscious competence. If I practice for years and get extremely good at this, I'll be able to play as a virtuoso, and simultaneously be able to think about other things. I will have reached the level of unconscious competence.*

This image is useful in suggesting that adding more lesson observation to the 15000 hours or more 'apprenticeship' is not likely to be fruitful until some first

hand experience of whole class teaching has been obtained. It also offers an explanation of why some class teachers find it difficult to talk about their own skills, having reached a level of unconscious competence.

Joram and Gabriele (ibid.) use an analogy that I also have found useful in preparing student teachers for their course and for helping them to see the relevance of both theory and practice. This is in drawing out comparisons between learning to teach and learning to drive. Initially I ask students to make the comparison, collect their thoughts in a whole class discussion and then add some ideas of my own. The collection of ideas might typically include:

1. Learning to drive involves a certain amount of theory (e.g. highway code) as well as practice in the driving seat. Learning to teach involves theory (e.g. about pupil development) and practice in the classroom.

2. There will need to be a shift of attention from the controls *inside* the car to what is going on *outside,* in comparison with an initial focus on teaching to a later focus on learning.

3. Whilst learning how to control the vehicle is a necessary part of learning to drive, it is the journeys which are the point of the exercise. Class control is important, but it is not an end in itself and it is ultimately more important to take the learning forward. Controlling a stationary car is of strictly limited value.

4. The need to automate procedures in driving such as "mirror-signal-manoeuvre" in comparison with routines in teaching such as "Pens down please, look at me everyone ... "

5. The need to develop multi-tasking skills, for example so that you can change gear, indicate and slow down simultaneously or the ability to write on the board, think about the next stage of the lesson, watch pupils' behaviour and be talking to the class all at the same time.

6. The need to give clear "signals" of intention, e.g. "In a minute, when I tell you to, I want you to...."

7. The ability to change pace or direction in a car, in comparison with the need to monitor and adjust your pace in teaching and sometimes the need to take an alternative approach to a topic.

8. For both learner drivers and student teachers, the need for supervision, for evidence of capability, for final "licensing".

It is difficult to be clear about what happens as a result of such discussion, other than I often find student teachers using the "driving" analogy in their own writing and in describing problems which have arisen in own classroom teaching. The image *appears* to be adopted by many students, supplanting their own initial images, but I have not yet sought to rigorously test this impression.

In dealing with class management issues, the use of images can be revealing. Early in my own career I was asked what it was like to manage difficult pupils. I replied that it was "Like lion-taming, only you are not allowed to use a whip or a chair." Since then I've heard others use this image and seen the TV programme 'In The Lion's Den'. The lion-taming image reveals a preoccupation with control, with viewing the classroom as a cage for both pupils and teacher, with trying to get pupils to do tricks, with fear, with confrontation. I offer student teachers alternative images to discuss such as: The Shepherd (a shift of emphasis to a caring role), The Sheep Dog (management through being in the right place at the right time with the right expression on your face!), and The Symphony Conductor (a facilitative and coordinating role). I invite students to describe or devise their own images, to be conscious of their images, and reflect on these over the year.

When introducing sessions for student teachers on working with low attaining pupils I use an image of The Poor Skier. In presenting this, students are invited to

> *Imagine what you would feel like if you had never skied before, but suddenly a law was passed requiring you to attend compulsory ski training three times a week. Week after week you duly turn up, but you are hopeless. Week after week for SIX YEARS all that you can do is to fall flat on your face. Now you are in Year 7, its time for The Big Slope! Are you filled with enthusiasm?*

The intention of the image is to engender some empathy for the plight of the low attaining pupil, to realise that motivation is likely to be an issue and to suggest that alternative approaches might be helpful ("Hey, let's go tobogganing this week!"), rather than pressing on with more of the same.

Some images I use in discussing the provision of differentiation to cater for pupils of varying abilities include the Motorway Model (three parallel strands of pupil work for varying speeds. or abilities) and The Hill:

> *In "The Hill" model pupils are presented with work of gradually increasing difficulty, and a fixed time to get as far as they can up The Hill as exemplified by the familiar graded exercise. The intention is to allow pupils to extend themselves to their own level of ability. In practice achievement is more determined by the rate of working rather than ability, as slower workers do not get far, whilst the keener rush on ahead. If this whole process is repeated over a number of lessons, it becomes the Hilly Model. (Smith, 1989, p. 9)*

This may enable students to see more clearly the drawbacks of habitually using the Hilly Model in their teaching, as some mathematics student teachers are inclined to do.

Conclusion

The intention was not to produce a fully rounded theoretical framework and a complete set of example images. I hope to have provided some examples of images and to have said enough about the theory to intrigue. I invite readers to try these and other images in their own work and to share these through professional journals and informal contacts, including email.

References

Joram, Elena and Gabriele, Anthony J (1998) Preservice teachers' prior beliefs: transforming obstacles into opportunities, *Teaching and Teacher Education*, 14, 2, pp.175-191.

Lortie, Dan C. (1975), *School Teacher: a sociological study*, University of Chicago Press, Chicago and London.

Kagan, Dona M. (1992), Professional Growth Among Preservice and Beginning Teachers, *Review of Educational Research*, 62, pp.129-169.

Smith, Jim (1989), Curriculum delivery models*, Mathematics Teaching*, 127, p.9.

Watzlawick, P. (1978) *The Language of Change,* New York: Basic Books.

Wubbels, Theo (1992), Taking account of student teachers' preconceptions, *Teaching and Teacher Education*, 8, 2, pp.137-149.

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Identifying and Dealing with Misconceptions and Errors in Primary Mathematics: Student teachers record their experiences

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The new requirements for initial teacher training have caused institutions to consider the content of their sessions and assignments. This article reflects on an attempt to meet those parts of the requirements dealing with errors and misconceptions by asking students to keep a diary of misconceptions which they observe while working in school and to write an essay based on the diary. In discussing the students' responses, we conclude that while parts of the requirements were satisfied by the assignment, other parts are more problematic. However we also conclude that the diaries had several other benefits and highlight overlap between areas which the requirements present as fragmented.

Introduction

This article discusses those parts of the Initial Teacher Training Requirements (DfEE 1997; DfEE1998) dealing with errors and misconceptions. The focus is on an attempt to meet the requirements through an assignment based on students keeping a diary of misconceptions which arose while they were working in school. The diaries were then handed in together with an accompanying essay as a marked assignment. Analysis was then carried out on a sample of the essays and diaries and we present examples of some of these as case studies, with commentary based on the analysis. In reflecting on the assignment we also draw on other information such as discussion with students, notes from tutors' meetings, student feedback forms and external examiners' reports.

Literature Review

The recently introduced requirements for initial teacher training include a detailed annex on primary mathematics (DfEE 1997; DfEE1998). Part of this annex deals with common errors and misconceptions in mathematics. Despite its current prominence, this is far from being a new issue. For example Shulman (1986) highlights the study of misconceptions as the point where research on teaching and learning coincide most closely. He regards knowledge of potential difficulties as being at the heart of required pedagogical knowledge.

The current requirements state that trainees must be taught how to recognise common pupil errors and misconceptions in mathematics, examples of which are given in the document. Trainees are also expected to understand how they arise and can be prevented, how to remedy them and how to avoid teaching in ways which contribute to or exacerbate misconceptions, with examples again given.

There are several inter-related aspects here with the first, recognising common errors and misconceptions, being apparently the most straightforward. Many of the books commonly used on teacher education courses consider the difficulties children may encounter in particular areas of mathematics (e.g. Duncan, 1992; Haylock and Cockburn, 1997). Teachers are also supplied annually with an analysis of children's difficulties as perceived through National Curriculum testing (e.g. QCA 1998a, 1998b). The distinction is not always drawn in the literature between misconceptions and other errors, nor is the difference clear in the teacher training requirements themselves. The problem of making distinctions between children's difficulties has also been remarked on by OFSTED (1994) who claim that teachers have difficulty in distinguishing between 'simple careless errors' and 'fundamental lack of understanding'

Possible explanations for children's difficulties in learning mathematics are also widely available. These include difficulties related to the language used (Donaldson, 1978) and to the presentation of abstract concepts to young children (Hughes, 1986). Cockburn (1999) looks at a range of sources of errors and subdivides them into three main categories. The overlap between these is acknowledged and each category is further sub-divided. Explanations linked to concept formation include those of Liebeck (1984) who discusses the issue of 'noise' during concept formation and Skemp (1971) who considers hierarchies of concepts. A more recent consideration of misconceptions suggests children sometimes over-generalise in their attempts to explain patterns that they notice (Askew and Wiliam 1995).

There is a clear connection between explanations for errors and suggested solutions for dealing with them, with the sources mentioned above all going on to suggest possible solutions linked to their explanations. As well as the connection between specific errors and solutions it is worth considering a more general link between teachers' views about why children make errors and their possible action. A recent study seeking to identify effective teachers of numeracy categorised teachers according to belief, one strand of which concerns belief about how pupils learn including their possible difficulties (Askew et al., 1997). One group was seen as putting pupil errors down to failure to grasp what has been taught and the remedy is therefore likely to consist of reinforcement of what is seen as the correct method. Another category saw mistakes as related to pupils not being ready for certain ideas. The most effective group, in contrast, saw the need to recognise, make explicit and work on pupil errors.

A consideration of how misconceptions arise leads to the issue of how, or to what extent, they can be prevented. In relating misconceptions to under or over-generalising, Askew and Wiliam (1995) suggest that choice of examples may help in reducing misconceptions, though they assert that it is not possible to avoid any misconceptions arising. They advocate an approach which involves constant exposure and discussion of misconceptions. This 'conflict' approach can be contrasted to a 'positive only' approach designed to anticipate and avoid misconceptions. A study comparing these two approaches found that while both were effective, the conflict approach was significantly more effective over a longer term. This study also found that there were no pupils who regressed when they were introduced to misconceptions that they themselves did not possess (Swan 1983).

Reviewing the literature suggests that consideration of children's difficulties is a familiar part of initial training and of teaching, though the distinction between errors and misconceptions is less straightforward. Understanding how misconceptions arise and how to remedy them are inter-related and more complex, with a range of explanations worthy of consideration. The prevention or avoidance of misconceptions is perhaps the most problematic area, with the existing literature suggesting this is a more complicated issue than the initial training regulations imply.

The Diaries and the Assignment

The 'Misconceptions Diary' was an idea first used with a cohort of primary initial training students in the academic year 1997/1998. These students were in the second year of a three year undergraduate course. They were introduced to the theme of misconceptions at the beginning of the year and asked to start

keeping a diary in which to record any misconceptions or errors they observed while working in school. A possible pro-forma was offered for the diary, but students were free to adapt this. Students were in school one day a week and keeping the diary was one of many directed tasks they had to complete in that time. While the diary was being kept, the theme of misconceptions was returned to occasionally in college sessions and it was also a sub-theme of much of the other work carried out during the year. For example, students running presentations on mathematical topics such as money or place value were expected to give consideration to possible misconceptions. After a term and a half of keeping the diary the students were required to submit it together with a related essay as a marked assignment. Guidance for the essay required them to start with a consideration of their observations, the actions taken and the outcomes and then move on to consider analysis of the errors or misconceptions and suggest long term strategies which may help to avoid the same problem occurring in the future.

The diary and assignment were used again in the academic year 1998/1999 with some modifications. These consisted mainly of more detailed guidance to students and schools, together with closer monitoring in the early stages. There were also increased opportunities to debate issues arising from the diaries in college sessions.

Diaries and Essays – Some Case Studies

A case study approach has been used in order to present a qualitative picture of the students' work and to show particular examples. Two main criteria were used in choosing the examples. The first criterion concerns the errors and misconceptions themselves, with diary pages being chosen which reflect fairly common examples found in two different age groups. The second criterion concerns the quality of the student's work and the issues they raise. Extreme cases have been avoided and the marks given for the assignment suggest that Tina's essay and diary were strong but not outstanding while Shirley's work was average. Both Tina and Shirley raised issues which were common to a number of students. In the commentary which follows attention has been drawn to aspects of these diaries and essays which are common to many students as well as to those which are more unusual. At some points reference has been made to the analysis of a group of diaries in order to give an idea of how widespread certain features were.

Case Study One – Tina

Tina was in a reception class. She found that many of the entries in her diary were concerned with counting errors and therefore chose to focus on this in her

essay. A page of Tina's diary is shown (figure 1) and one entry concerns Suzie who missed out objects while counting. In her essay, Tina identified this as a problem with one-one correspondence which she went on to discuss in some detail making use of background reading. She also talked in more detail about the action she took and Suzie's response.

> *I tried to encourage Suzie to touch each object as she counted. This seemed to work with numbers below 5. When I put out 7 objects and asked her to count them, she struggled because she was more concerned with trying to remember the numbers that come after 5 rather than the amount of objects. Therefore, I feel Suzie isn't yet confident enough to use numbers above 5. To help Suzie develop the concept of one-one correspondence, I feel she needs. more practice counting objects up to numbers 5/6 and I would encourage Suzie to try and separate the objects as she has counted them.*

There are features of Tina's work which are common to students in reception classes. For example many of these students decided to write mainly about difficulties in counting. Their work was generally well supported by background reading and their own observations were usually in agreement with the reading. Another common feature was the ability to move on from analysis of errors to identify appropriate tasks for that child to do next. Tina, in common with about three-quarters of the students in her cohort suggested activities which could be used as a next step with the child concerned. In contrast about half the students made suggestions about ways of avoiding or reducing the problem in the future, though many of these suggestions were very vague. Tina was one of those who did not offer ways of avoiding the problem, something she had ruled out in reviewing background reading in the first page of her essay.

There were aspects of Tina's work which were strong in comparison to some students. One strength lay in the quality of her observations, supporting a point made by an external examiner who, having read a sample of essays said the quality of essay depended on the quality of observation. For example Tina was only able to distinguish between Suzie's difficulty and Ruth's (see figure 1) because she had observed the children while they were actually counting rather than just looking at the outcome of the task. Tina was also more detailed than many students in her observation of the children's responses, something which had an impact on her recorded outcomes, which acknowledged that the difficulties had not been sorted out instantly.

50

Case Study Two – Shirley

Shirley was in a year two class. Looking at a page of Shirley's diary (figure 2) suggests some contrasts with Tina's, the most obvious being the mention of work books in the year two class in contrast to the practical activities in the reception class. This pattern was broadly reflected across all the groups of lower primary students. Just over half the lower primary students used written evidence of errors or misconceptions, though this included all the students working with year two. Students in reception tended to use written evidence mainly if the difficulty was related to recording or drawing; for example problems with number formation or drawing shapes. Students in year two, in contrast, often used written evidence of calculations and also, like Shirley, sometimes included evidence gained when they were asked to help children who had been working alone through work books and become stuck or made errors.

Child.	Activity.	Problem.	Action.	Outcome.	Strategy
Darren	Naming different shapes. Square /circle / triangle.	Darren called a triangle a square.	Asked him why the triangle was a square – he said because it was straight sides. I took the two shapes and allowed him to touch and look at the differences.	Still a little unsure	sorting shapes – triangles – squares variety of shapes. use the long in context eg. a square book.
Suzie	Match a number of objects with the correct numeral.	Suzie misses out several objects when she counts.	Encourage her to touch / mark each object counted.	Alright with numbers below 5.	counting numbers above 5 in order. seperate objects as you count them.
Ruth	Match a number of objects with the correct numeral.	Ruth counts some objects more than once.	Encourage her to touch / move each object.	Still occasionally touched object twice.	Put objects in a line, clear short read. .....

figure 1

INDIVIDUAL/ GROUP	OBSERVATION	ACTION	OUTCOME
Stephanie (year 2)	Working in maths book, SPMG, level 2, book 3. Working on money problems. Needed to work out what coins were needed to make 21p, 24p, 33p, 45p and 50p - each being that they counted in tens. For 21p Stephanie thought that she needed 2p and up. Stephanie said she needed for 24p that she needed a 2p and a 4p.	To help Stephanie I needed to set/up the up to work out what was left. 2cp. Asked how so would make. That and then uncovered the up to show what to make 21p, and discovered a 1p did the same with 24+p.	After having problems with the first one, Stephanie seemed to grasp the idea. She needed a little help with the next one, but managed the last and without any help
Keith (year 2)	Working in SPMG book. Difference in price. 9 - 6 = Keith could work out the sum at the end because it was a 0 straight forward sum. He was unable to work out what the difference was. He didn't understand what difference meant. Keith thought it was really complicated.	Explained to Keith that find the difference means take away. This took a long time to explain. Worked through a couple of examples with Keith.	Keith seemed to understand in the end after a big explanation. Needed to work through examples, but Keith managed to complete the rest of the sums in his book.

figure 2

Both of the entries shown from Shirley's diary are of fairly common difficulties and there are also aspects of Shirley's responses which were common to many students. On the whole students were confident in giving immediate help to children experiencing difficulties in the way that Shirley helped Stephanie and Keith. In almost all cases the help consisted of working with the individual concerned and talking to them about the difficulty. Use of questions, as in Shirley's response to Stephanie, often formed part of the strategy. Students also frequently stayed with the child while they did a few more examples or returned later to see how they had coped, a strategy Shirley used with both Stephanie and Keith.

Keith's difficulty with the word difference (see figure 2) was one observed by many students. All but one of these put the difficulty down to the language, not the calculation itself. Shirley, unlike most of the other students, presented evidence for this decision in noting that Keith could complete 9 - 6 = , but could not record the difference between 9 and 6. Many students, like Shirley, identified language as a source of difficulty, with this being the most frequent single explanation amongst the lower primary students and the major theme of many of their essays. It could be seen as ironic that while acknowledging the difficulties caused by language in mathematics, Shirley in fact misuses the word 'sum' herself. However, although many students saw language as a difficulty they did not all see use of correct mathematical language as the solution, with some feeling alternatives were more appropriate for young children.

Students' Reaction

On the whole the reaction of the students was positive following some initial anxieties. In the first year in particular there were concerns about the practicalities of how students would go about trying to identify and record misconceptions and errors. Some students expressed anxiety because, despite sessions on the subject, they were not really sure what a misconception was. This was not helped when some reported that the class teachers they were working with had 'never heard the word misconception', by which it was presumed that they meant the teachers had not heard the word in an educational context.

As the assignment got underway and concerns were discussed in sessions and examples given, the majority of the students were able to make a start on the assignment. For a minority the question of access to and involvement in mathematics sessions in schools was still an issue. For the majority the focus now shifted to how to deal with particular difficulties and to the identification of particular background reading.

Once the assignments were completed most students were positive about what they had learned from it. A very few students in the first cohort felt they had not come to grips with the diary, mostly because they had not become involved in mathematics teaching. Interestingly a few students who wrote very strong essays also expressed the view that identifying and dealing with misconceptions had proved difficult as they regarded the issues involved as complex and problematic. There were also some concerns about workload, mainly from the first cohort and about access to books. However comments from students confirmed that the majority felt they had gained from completing the diary and assignment.

> *I have found many of the misconceptions I discovered interesting to investigate. I also agree that careful explanation, discussion and interactions with the children are the best way to tackle and overcome misconceptions in the classroom.* Elaine

> *Therefore the teacher needs. to challenge and question in order to clarify if any misconceptions are being constructed, it is only then that positive or pre-emptive strategies can be applied.* Jo

> *This assignment has alerted me to the fact that we are asking a lot of young children when expecting them to understand money.* Lorraine

What is interesting about the comments made by these and other students is that they have talked not just about the act of collecting diary entries but about the issues which this had raised for them and the conclusions they have reached about future action. Thus, while Lorraine has developed an awareness of the difficulties children may have with a particular aspect of mathematics, Jo has thought about the need to 'challenge and question' in order to uncover misconceptions. However, Elaine has focussed on 'careful explanation, discussion and interactions with the children' in dealing with misconceptions.

Tutors' Reaction

There were ways in which the tutors' perception of the diary and assignment seemed to match that of the students, as well as areas where views differed. The tutors acknowledged that involvement with mathematics teaching and knowledge of the children were required if students were to complete the diary and assignment to a high standard. However, whereas some students saw this as problematic the tutors saw it as an important ingredient of the assignment and made no apologies for it. They felt that student involvement with mathematics teaching was an essential part of time in school and the fact that the assignment demanded this was one of its strengths. Tutors did acknowledge however that some students needed assistance in going about this. Thus the guidance to students on how they might agree access to mathematics

with their teachers was increased for the second cohort of students. The diaries and assignment were the subject of discussions between tutors and teachers at training days which meant changes could be made to the workload for students and to the information given to teachers.

Although the tutors were positive about the work of the first cohort of students they felt improvements could be made in several areas and these were implemented in the second year. Like the students the tutors felt more structure and guidance would be helpful as well as subtle monitoring to ensure that students had all made a start on their diaries and on considering possible action at a relatively early stage of the course. Thus in the second year there was more explicit discussion of the diaries in college sessions, with opportunities taken to share examples and possible solutions and to talk through issues arising in some detail. In particular tutors took the opportunity to discuss what action students might take if they were not aware of any misconceptions, for example in use of probing questions or choice of varied examples.

Whereas the majority of students felt they had learned about the identification and remediation of misconceptions, the tutors felt that many students had a significant amount to learn in these areas. In particular they felt many students had not really engaged with the idea of what differentiates an error from a misconception. In some cases trying to use these two categories had even proved unhelpful. This was because some students drew a stark distinction between misconceptions, due to misunderstanding and requiring action and 'just errors', due to carelessness and not requiring action. One such example was a student who identified a systematic subtraction bug and discussed at some length whether or not it was a misconception. Having decided it was 'just an error', led her to conclude that the child was simply being careless and thus no action was taken. Another area leaving considerable room for improvement was relating the explanation of errors or misconceptions to students' background reading and their understanding of how children learn.

Despite these reservations the tutors felt that all students had gained something from the diary and the assignment. In this respect tutors' comments were consistent with the student comments quoted above in acknowledging that the issue of pupil difficulties is related to many other fundamental issues in teaching. For example there were instances where the need to keep the diary provided a focus for students who needed more guidance on how and what to observe in the classroom. The fact that the students had to act on their diary entries also led them to reflect on the usefulness of what they had written and in many cases entries became more specific as the diary progressed. For some the diary also acted as an ongoing record of their work with particular children and of their increasing involvement in the classroom. In taking action to help

children having difficulties the students had to be precise in their learning objectives and this was evident in some of the plans for follow up activities submitted with the assignments. Also evident was the detail in some accompanying evaluations and in particular the fact that students took more care in considering whether their learning objectives had been met. It is clearly problematic to try to show to what extent any of these things were linked to and caused by the diaries. What is clearer is that the diary is a potential catalyst in moving students forward in all these areas and in raising discussions about a wide range of issues.

Conclusion

In the view of both students and tutors the diary and related assignment went some way in meeting the relevant requirements. In particular students increased their awareness of common misconceptions in mathematics and all had experience in identifying and trying to overcome errors and misconceptions.

In the view of the course tutors many students still had some way to go in their understanding of some of the issues and in relating their classroom experience to their reading. In particular many students found it hard to engage with the issue of the difference between a misconception and an error. Another area of weakness was in considering long term action to reduce or avoid misconceptions. Tutors are continuing to consider how the assignment and the college input can be refined to improve these aspects, but the question is also raised about whether there are issues here which may not be solved and whether things are more problematic than the teacher training regulations imply.

Both the students and tutors recognised that the work on misconceptions had many benefits in addressing areas such as planning, classroom observation and evaluation. In doing this it highlighted the inter-relationships between areas and thus avoided the fragmented and itemised approach that the initial training regulations could be seen to encourage. The intention is to retain and refine the diary and assignment for use in future years.

References

Askew, M. and Wiliam, D. (1995) *Recent Research in Mathematics Education 5-16*, London: HMSO.
Askew, M., Brown, M., Rhodes, V., Johnson, D. and Wiliam, D. (1997) *Effective Teachers of Numeracy*, London: King's College.
Cockburn, A. D. (1999) *Teaching Mathematics with Insight*, London: Falmer Press.

DfEE (1997) *Teaching: High Status, High Standards*, Requirements for Courses of Initial Teacher Training, Circular Number 10/97, London: DfEE.

DfEE (1998) *Teaching: High Status, High Standards*, Requirements for Courses of Initial Teacher Training, Circular Number 4/98, London: DfEE.

Donaldson, M. (1978) *Children's Minds*, London: Fontana Press.

Duncan, A. (1992) *What Primary Teachers Should Know About Maths*, London: Hodder and Stoughton.

Haylock, D. and Cockburn, A. (1997) *Understanding Mathematics in the Lower Primary Years*, London: Paul Chapman Publishing.

Hughes, M. (1986) *Children and Number: Difficulties in Learning Mathematics*, Oxford: Blackwell.

Liebeck, P. (1984) *How Children Learn Mathematics*, Middlesex: Penguin.

OFSTED (1994) *Science and Mathematics in Schools: A Review*, London:HMSO.

QCA (1998a) *Standards at key stage 1, English and Mathematics*, London: QCA.

QCA (1998b) *Standards at key stage 2, English, Mathematics and Science*, London: QCA.

Shulman, L. S. (1986) Those who Understand: Knowledge Growth in Teaching, *Educational Researcher*, 15, 2, pp.4-14.

Skemp, R. (1971) *The Psychology of Learning Mathematics*, Middlesex: Penguin.

Swan, M. (1983) *Teaching Decimal Place Value, A Comparitive Study of "Conflict" and "Positive Only" Approaches*, Nottingham: Shell Centre.

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MER11 Editorial
Epilogue: Extracting the generality

In the second part of this editorial we briefly explore some of the generalities of practice described in these papers. We see this as the beginning of a debate and welcome responses. As described on page 5 we are looking to discover aspects of practice in terms of the practical wisdom, professional traditions and the ways that these interrelate with teacher-educators' subject knowledge to create sessions for the student teachers.

The first thing that cries out from all the articles is the very complexity of engaging in thinking about teaching and learning and mathematics. Each of the articles describes explicitly the complexity of learning to be a teacher, something that the normalising of ITE into standards and syllabuses and quick entry routes does not include. The lovely quote from Lortie, 1975 (page 40), offers a reminder that teaching is not just a body of craft knowledge to be observed and absorbed but that becoming a teacher demands great intellectual effort - that you may have or gain learner-knowledge but of itself is not enough when planning for teaching (Proof and Cabri); that I might tell you about misconceptions, errors and difficulties but noticing them and noticing the difference between them is an art to be learnt with guidance; that I might experience empathy for the low-attainer in not being able to learn to ski but translating that into the classroom requires other skills; that I might help you to recapture the sense of awe and wonder about learning mathematics, but will the examples sustain?   Making meaning about teaching and learning and mathematics is in the layered teaching of all the authors, via the discussion, the activity and the writing of the student teachers.

Expecting generality is crucial to the practice of all educators.   It is the generalisation that allows transferability of knowledge from a particular situation in which something is practised to many other situations in which similar practice may occur. In our teaching we have to use specific examples in the hope that they are seen by our students as generic.

> *A generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general. This is done by means of stressing and ignoring various key features, of attempting to structure one's perception of it. Different ways of seeing lead to different ways of knowing.   (Mason and Pimm, 1984, p.287)*

In mathematics teaching this is prevalent in the 'worked example' and, in relation to the learning of mathematics Mason and Pimm offer the reminder that:

> *Unfortunately it is almost impossible to tell whether someone is stressing and ignoring in the same way you are.*

So too in teacher education especially for the shorter routes into teaching (PGCE, SCITT, new Graduate entry schemes) whose students spend more time in the 'doing' mode than the 'reflecting' mode. Observing practice to understand the practice is different from observing to copy that practice and as Cathy Smith reminds us there is a "disadvantage of attempting both together in the lack of distance".

Earlier to the previous quote in their paper on generic examples Mason and Pimm (1984) state the following:

> *... A teacher having written an example of a technique on the board, is seeing the generality embodied in the example, and may well never think of indicating the scope of the example, nor of stressing the parts that need to be stressed in order to appreciate the exampleness. However, the pupils have far less experience, even with the particular instance of the discussion (and may well be unaware that there are others) which as a consequence absorbs all their attention. The pupils may only see the particular (which is possibly for them quite general, i.e. not mastered). As a result they often try to 'learn the example'. (p.286)*

We are reminded therefore of two things. Firstly, an implication for our own practice, that student teachers might be seeing something different from that intended by the session or in fact might not be able to see the generality, only 'learn the example' (for example use the five questions in the proof session in the given form in the classroom). Secondly, that in working with these case studies we are aware that different readers will notice and extract a different generality. In our reading of these case studies of ways of working with ITE students we wish to focus on our understanding of the practice described interpreted in terms of the model presented in the prologue. We have taken aspects of the model and begun to exemplify each of them in what follows.

Mathematics teacher-educator as mathematics teacher

The Cabri and Proof articles show the mathematics educator acting as a mathematics teacher, the aim in each case being to enhance the learner-knowledge of the students. Encouraging the students to reflect upon the learner knowledge is evident in both cases but with different urgency according to the 'speed' of each course - the Cabri session is created with first year BEd students and in the Proof session we are working with one year PGCE students. Interestingly the Cabri session takes one of the professional traditions of the mathematics educator (that of the reflective practitioner, Schon) and explicitly

uses this approach in this geometry session "critical reflection is an essential part of the development from mathematical activity per se to knowledge about mathematics".

Both articles describe activities deliberately set up to challenge the learners' pre-conceptions of the mathematics - the proof questions deliberately lead down the algebra path, the Cabri session starts with construction of squares to connect to special case definitions from different quadrilaterals. This practical wisdom (that learner-knowledge itself needs. challenging) is paralleled in the other papers which describe sessions created to challenge and change preconceptions about teaching and learning mathematics.

Professional traditions

In all the papers authority for actions is gained from the literature and research used to support the sessions (e.g. HMSO 1985; Swan 1983; Askew 1997 etc.). Anne Watson reasons that

it is important to spend time on authoritative support of student sensitivities because when learners find themselves voicing opinions which contradict current orthodoxy they may need more than reassurance of a teacher or a tutor to believe that their views are worthwhile.

We would add to this that sometimes tutors' alternative views need support from elsewhere when challenging existing orthodoxies.

The role of the reflective practitioner, mentioned earlier, is evident in all the articles each of which captures a different instance. Developing the student as a researcher in his/her own classroom, asking for reflection upon remembered learning, upon new learning or upon new images created in a session, using reflection as a technique to encourage diary writing, all of these instances offer ways in which we work with students.

The starting points for Wonder and Misconceptions were explicitly placed in the new professional traditions for ITE, in the Standards (Circular 4/98). This has direct parallels with a mathematics teacher accounting for choices in planning with coverage of a syllabus or the National Curriculum. Time will tell how this new level of accountability will affect practices.

Practical wisdom

One of the major features of practical wisdom in these articles was the 'connecting' that happens in each of the sessions, although these are described

differently and represent different levels of connections. This has direct parallels with a mathematics teacher connecting across and within the mathematical content knowledge (Askew's 'connectionist teacher', our definition of teacher-knowledge). For example, the different layers of working explicitly described in Cabri and Wonder; working with the mathematics, about the mathematics and about learning; in Proof working on connecting between learning mathematics yourself and how others might come to learn mathematics; in Misconceptions about relating theory to classroom practice to the work of different attainment in the student teachers.

The next practice evident in the papers was the level of challenge to the student teacher's preconceptions of teaching and learning mathematics and with that challenge the authors bring a sense of their working to promote change. In Cabri and Proof the challenge was to the learners' own subject knowledge, in Images and Misconceptions to the students' sense of being a learner and the different ways that individuals come to learn. This aspect of practical wisdom also has its roots in the professional traditions of the teacher-educator when Jim Smith quotes from research that "targeting student teachers' imagery has significant impact in changing preconceptions".

Other points we noticed also had direct parallels with teaching mathematics: in Misconceptions the use of assignments to give value to the activity and to the learning, though here the authors offer a way of working on subject knowledge through developing a diary - something not commonly seen in classrooms; in Images the use of empathy to extend the students' experiences with the use of particular words being carried forward to other sessions, to say a word that carries with it a shared meaning which will immediately access shared understanding. For example we might say "Square Roots" to the students which recreates images of learning algorithms, the Images article might allow the author to say "Skier" which might recreate understanding about pupils with learning difficulties.

To conclude

In identifying the points above, we realise that we are revealing more of our own beliefs than those of the authors. Working on these papers reminds us of the need to revisit earlier descriptions of our practice as teacher-educators. Deliberate reflection on such case studies may enable us to identify, initially for ourselves, the boundaries of our subject knowledge and to articulate practical wisdom and the ways in which we work with and account for professional traditions, old and new.

References

Mason, J. and Pimm, D. (1984) Generic examples: seeing the general in the particular, *Educational Studies in Mathematics*, 15, pp.277-289.